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On the determination of the basin of attraction for stationary and periodic movements.
Fast motions in biomechanics and robotics, 147–166, *Lecture Notes in Control and Inform. Sci.*, 340, *Springer, Berlin*, 2006.

Techniques for estimating the region of attraction of stationary and periodic movements are reviewed, especially those that can be found in human coordinated movements. The mathematical model used is based on a muscle-skeletal system without reflexes derived using simple modeling assumptions. It yields a second-order ordinary differential equation of the following form:

$$\dot{\beta} = \omega\dot{\omega} = f(t, \beta, \omega),$$

where ω is the angular velocity. Two typical muscle-skeletal movements are analysed using the above framework, namely (i) standing, where the force-length function $f(\beta, \omega)$ is independent from t, and (ii) walking, where the solution of the above equation has a solution which is periodic with period T (i.e., $f(t + T, \beta, \omega) = f(t, \beta, \omega)$). The force function f has the particularity that $\frac{\partial f}{\partial \omega} f(t, \beta, \omega) < 0$, because of muscle-force considerations.

The basin of attraction is then estimated for each class of behaviors. For case (i), the linearization furnishes a system matrix from which a local Lyapunov function can be obtained. However, after taking full advantage of the dynamical structure, and particularly $\frac{\partial f}{\omega}(\beta, \omega) < 0$, the special Lyapunov function

$$V(\beta, \omega) = -\int_0^{\beta} f(\tilde{\beta}, 0)d\tilde{\beta} + \frac{1}{2}\omega^2$$

leads to a subset of the basin of attraction. Consider $(\beta_0, 0)$ and $(\beta_1, 0)$ to be some equilibria, with $\beta_i \in (0, \pi)$, $i = 0, 1$; then set

$$S = \{(\beta, \omega) \mid V(\beta, \omega) < V(\beta_1, 0)\} \cap (0, \pi) \times \mathbb{R}.$$

Now, if the closure $\overline{S}$ is connected and compact and $f(\beta, 0) \neq 0$ for $(\beta, 0) \in S \setminus \{(\beta_0, 0)\}$, the set S is included in the attraction domain of β_0, $S \subset \mathcal{A}(\beta_0, 0)$.

Another technique for estimating the region for point stabilization (standing) is to resort to radial basis functions through setting the Lyapunov function to

$$V(x) = \sum_{j=1}^{N} \alpha_j \langle \nabla_y \Psi(x - y), F(x_j) \rangle\big|_{y=x_j}$$

where $\Psi(x) = \psi(\|x\|)$ is a fixed radial basis function and F stands for $(\omega f(\beta, \omega))^T$. The coefficients α_j are chosen such that v satisfies the equation on the grid. The explicit solution of α_j is easily achieved by solving a system of linear equations.

In the case of walking, periodic movements lead to a more complex analysis. Floquet theory transforms the initial time-varying system $\dot{x} = F(x, t)$ into $\dot{y} = G(t, y)$ after setting $(t, y) :=$

$(t, x - \tilde{x}(t))$. This transforms the periodic solution $\tilde{x}(t)$ into the zero solution $y(t) = 0$. For example, for a system leading to the particular structure $\dot{y} = G(t)y$, each fundamental matrix $X(t)$ has a representation $X(t) = P(t)e^{Bt}$ where $P(t+T) = P(t)$ is a periodic (2 ×2) matrix-valued function and B a (2 × 2) matrix. The eigenvalues of B are called Floquet exponents, the real parts of which indicate asymptotical stability as long as they are all negative. A radial basis function can also be used after adding the differential equation $\dot{t} = 1$ to $\dot{y} = G(t, y)$ and using the non-autonomous technique.

Unfortunately, Lyapunov functions and Floquet theory need the knowledge of the periodic movement $\tilde{x}(t)$. This is not the case with Borg's method, which is based on the following notion of a Riemannian metric. Consider a matrix-valued function $M(t, x)$ which is symmetric and positive definite for each $(t, x) \in S^1_T \times \mathbb{R}^2$. Now consider $M'(t, x)$ to be the matrix with entries $m_{ij} = \frac{\partial M_{ij}(t,x)}{t} + \sum_{k=1}^{2} \frac{\partial M_{ij}(t,x)}{x_k} F_k(t, x)$ (the orbital derivative of M), so that after defining

$$L_M(t, x) := \max_{w^T M w = 1} \left[M D_x F(t, x) + \frac{1}{2} M^{'}(t, x) \right] w,$$

the condition $L_M(t, x) < 0$ implies that there exists one and only one periodic orbit which is exponentially stable. Moreover, all points satisfying this inequality and belonging to a connected, compact, and positively invariant set also belong to the basin of attraction.

The paper considers the above techniques applied to the periodic movement of an elbow-joint and a knee-joint for which $f(t, \beta, \omega)$ is explicitly given, based on models of extensor and flexor muscles. The stability property is strongly dependent on the positive slope of the force-length function (elbow) and the moving center of rotation (knee-joint). A high co-activation also stabilizes the system. The stability of stationary movements depends on the position of the joint angle: for the elbow, small angles are stable, whereas large angles are unstable. For periodic movements, general answers to these questions are difficult to obtain.

{For the entire collection see MR2271880 (2007f:70005)}

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