A fast algorithm to remove proper and homogenous pairs of cliques (while preserving some graph invariants)

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Abstract

We introduce a family of reductions for removing proper and homogeneous pairs of cliques from a graph G. This family generalizes some routines presented in the literature, mostly in the context of claw-free graphs. These reductions can be embedded in a simple algorithm that in at most $|E(G)|$ steps builds a new graph G' without proper and homogeneous pairs of cliques, and such that G and G' agree on the value of some relevant invariant (or property).

Keywords: Proper and homogeneous pairs of cliques; Reductions; Graph invariants.

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1. Introduction

A pair of vertex-disjoint cliques $\{K_1, K_2\}$ is *homogeneous* if every vertex that is neither in K_1, nor in K_2 is either adjacent to all vertices from K_1, or non-adjacent to all of them, and similarly for K_2. Homogeneous pairs of cliques were first defined in the context of bull-free graphs [6], and seem to play a non-trivial role in combinatorial, structural and polyhedral properties of claw-free graphs. For instance, a well-known decomposition result by Chudnovsky and Seymour is as follows:

Theorem 1. *[4] For every connected claw-free graph G with $\alpha(G) \geq 4$, if G does not admit a 1-join and there is no homogeneous pair of cliques in G, then either G is a circular interval graph, or G is a composition of linear interval strips, XX-strips, and antihat strips.*

See [4] for the definition of graphs and operations involved in Theorem 1: we skip them, since they are of no use for the present paper. What is interesting to us is the fact that homogeneous pair of cliques are somehow an *annoying* structure: as it is written in [4], "There is also a ”fuzzy” version of this (*i.e. Theorem 1*), without the hypothesis that there is no homogeneous pair of cliques in G, but it is quite complicated". (This more complex version of the theorem is actually given in [5].) A similar situation can be found in the structure theorem on Berge graphs [3].

In the literature, some effort has been devoted to design *reduction techniques* to get rid of homogeneous pairs of cliques that are also *proper*. We say that a pair of cliques $\{K_1, K_2\}$ is *proper* if each vertex in K_1 is neither complete nor anticomplete to K_2, and each vertex in K_2 is neither complete nor anticomplete to K_1. Those reduction techniques are designed to preserve graph invariants, such as chromatic number [8, 10] and stability number [12], or graph properties, such as the property of a graph of being quasi-line [2], fuzzy circular interval [13], or even facets of the stable set polytope [7]. The state of the art complexity for recognizing whether a graph $G(V, E)$ has some proper and homogeneous pairs of cliques is $O(|V(G)|^2|E(G)|)$ [10, 14].

In this paper, we introduce a reduction operation that generalizes and unifies those different techniques. It essentially replaces a proper and homogeneous pair of cliques $\{K_1, K_2\}$ with another pair of cliques $\{A_1, A_2\}$ that is homogeneous but non-proper. A large number of pairs $\{A_1, A_2\}$ can be used in our reduction, and the choice of a particular pair is done depending on some invariant (or property) we want the reduction to preserve. Regardless of this choice and of the number of proper and homo-

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geneous clique of the input graph G, we show that our reduction can be embedded in a fast algorithm that iteratively replaces a proper and homogeneous pair of cliques $\{K_1^i, K_2^i\}$ with a non-proper and homogeneous one $\{A_1^i, A_2^i\}$, and outputs after $|E(G)|$ iterations a graph without proper and homogeneous pairs of cliques. We stress that the algorithm is not graph-class specific, i.e. it works with *any* simple graph in input. Our main result will be then the following:

Theorem 2. *Let $G(V, E)$ be a graph. Algorithm 2 builds a sequence of graphs $G = G^0, G^1, \ldots, G^q$, with $q \leq |E(G)|$, such that G^q has no proper and homogeneous pairs of cliques, and each G^i, $i < q$, is obtained from G^{i-1} by replacing a proper and homogeneous pair of cliques $\{K_1^i, K_2^i\}$ with an homogeneous pair of cliques $\{A_1^i, A_2^i\}$. The algorithm can be implemented as to run in $O(|V(G)|^2|E(G)| + \sum_{i=1}^{q} p(i))$-time, if, for $i = 1, \ldots, q$, it takes $p(i)$-time to generate $G^{i+1}[A_1^i \cup A_2^i]$, from the knowledge of G^i, K_1^i and K_2^i.*

Combining this theorem with a few results from the literature, we will show some more facts, among which:

- we can reduce in time $O(|V(G)|^{\frac{5}{2}}|E(G)|)$ the coloring problem (resp. the maximum clique problem) on a graph $G(V, E)$ to the same problem on a graph G' without proper and homogeneous pairs of cliques;
- we can reduce in time $O(|V(G)|^2|E(G)|)$ the maximum weighted stable set problem on a graph $G(V, E)$ to the same problem on a graph G' without proper and homogeneous pairs of cliques.

2. Preliminaries

Given a simple graph $G(V, E)$, let $n = |V(G)|$ and $m = |E(G)|$. We denote by uv an edge of G, while we denote by $\{u, v\}$ a pair of vertices $u, v \in V$. For a given $x \in V$, the *neighborhood* $N(x)$ is the set of vertices $\{v \in V : xv \in E\}$. We say that v is *universal* to $u \in V$ if v is adjacent to u and to every vertex in $N(u) \setminus \{v\}$. Let $S \subset V$, then $x \notin S$ is *complete* (resp. *anticomplete*) to S in G if $S \cap N(x) = S$ (resp. $S \cap N(x) = \emptyset$). Finally, we denote by $G[U]$ the subgraph induced on G by $U \subseteq V$; a C_4 is an induced chordless cycle on four vertices.

Definition 3. *Let G be a graph and $\{K_1, K_2\}$ be a pair of non-empty and vertex-disjoint cliques. The pair $\{K_1, K_2\}$ is* homogeneous *if each vertex $z \notin (K_1 \cup K_2)$ is either complete or anti-complete to K_1 and either complete or anti-complete to K_2.*

Definition 4. *Let K be a clique of a graph G and let $v \notin K$. v is* proper *to K if v is neither complete nor anti-complete to K, and $P(K)$ is the set of vertices that are proper to K.*

Definition 5. *Let G be a graph and $\{K_1, K_2\}$ be a pair of non-empty and vertex-disjoint cliques. The pair $\{K_1, K_2\}$ is* proper *if each vertex $u \in K_1$ (K_2, respectively) is proper to K_2 (K_1). A pair of vertex-disjoint cliques that are proper and homogeneous is also called a* PH pair.

We skip the simple proof of the following lemma.

Lemma 6. *Let G be a graph and $\{K_1, K_2\}$ be a homogeneous pair of cliques. Then $\{K_1, K_2\}$ is proper if an only if, for each $i \in \{1, 2\}$ and $x \in K_i$, there exist $y_1, y_2 \in K_i$ (possibly $y_1 = y_2$) such that x is non-universal to y_1 and y_2 is non-universal to x.*

In fact, one can show that for each clique K_i of a proper pair $\{K_1, K_2\}$ there always exist two vertices $x, y \in K_i$ that are non-universal to each other. Namely, we have the following (see Lemma 1 in [7]):

Lemma 7. *Let $\{K_1, K_2\}$ be a proper pair of cliques in a graph G. Then $G[K_1 \cup K_2]$ contains C_4 as an induced subgraph.*

Hence, when looking for a PH pair in a graph, one can start from a pair of vertices that are adjacent and not universal to each other, and then determine whether they have a *PH-embedding*, namely:

Definition 8. *Let u and v be two adjacent vertices of a graph G. We say that u and v have a* PH-embedding *if they are not universal to each other, and there exists a PH pair of cliques $\{K_1, K_2\}$ such that $u, v \in K_1$. We also denote by $PH(G)$ the set of pairs of vertices of G that have a PH-embedding.*

The next lemma is therefore trivial.

Lemma 9. *If no pair of vertices of G have a PH-embedding, then G has no PH pairs of cliques.*

Given two adjacent vertices that are non-universal to each other, a simple algorithm recognizes in $O(n^2)$-time whether they have a PH-embedding. This routine, which we report below, was independently proposed by King and Reed [10] and Pietropaoli [14] (see also [13]). Actually King and Reed designed an algorithm for a slightly different problem: call $\{K_1, K_2\}$ a *non-trivial homogeneous* (NTH) pair of cliques in G if $\{K_1, K_2\}$ is a homogeneous pair of cliques in G, and $G[K_1 \cup K_2]$ has an induced C_4. Lemma 7 implies that each PH pair of cliques is a NTH pair of cliques, and one can immediately check that the converse does not always hold. But given a NTH pair of cliques $\{K_1, K_2\}$, one can obtain a PH pair of cliques H_1, H_2 with $H_1 \subseteq K_1$, $H_2 \subseteq K_2$, by iteratively removing from $\{K_1, K_2\}$ vertices that are non-proper to the opposite clique. Thus, in order to find a NTH pair one can look for a PH pair: this is exactly what King and Reed do in [10] (see Section 3).

Algorithm 1 Finding a PH-embedding

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Require: A graph $G$, and a pair of adjacent vertices $\{u, v\}$ that are not universal to each other.
Ensure: A PH-embedding $\{K', K\}$ for $\{u, v\}$, if any.
$K' := \{u, v\}$; $K := P(\{u, v\})$;
while $K$ is a clique and $P(K) \neq K'$ do
    $K' := K$, $K := P(K)$;
end while
if $K$ is not a clique then there is no PH-embedding for $\{u, v\}$: stop.
else $P(K) = K'$ and $\{K, K'\}$ is a PH-embedding for $\{u, v\}$: stop.
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Theorem 10. *[10], [14] It is possible to implement Algorithm 1 as to run in* $O(|V(G)|^2)$.

Besides considering pairs of cliques that are proper and homogeneous, we will also consider pairs of cliques that are homogeneous but non-proper. This leads to the following definition:

Definition 11. *Let* G *be a graph and* $\{A_1, A_2\}$ *be a pair of non-empty and vertex-disjoint cliques that are not complete to each other. The pair* $\{A_1, A_2\}$ *is* C_4^{free} *if* $G[A_1 \cup A_2]$ *has no induced* C_4. *A pair of cliques that is* C_4^{free} *and homogeneous is also called a* C_4^{free}H pair.

It follows from Lemma 7 that no pair of C_4^{free} cliques is proper. We skip the simple proof of the next lemma.

Lemma 12. *Let* G *be a graph and* $\{A_1, A_2\}$ *be a pair of non-empty and vertex-disjoint cliques that are not complete to each other. Then* $\{A_1, A_2\}$ *is* C_4^{free} *if and only if the following holds: if* u *and* $v \in A_1$ *then* u *is universal to* v *or* v *is universal to* u *(note that this property holds if and only if the same happens with the vertices of* A_2*).*

The next lemma analyzes the possible intersections between PH and C_4^{free}H pairs of cliques.

Lemma 13. *Let* $G(V, E)$ *be a graph with a PH pair of cliques* $\{K_1, K_2\}$ *and a* C_4^{free}*H pair of cliques* $\{A_1, A_2\}$. *Then* $K_1 \cap A_2 = K_2 \cap A_1 = \emptyset$ *or* $K_1 \cap A_1 = K_2 \cap A_2 = \emptyset$.

Proof. We start with the following:

Claim 1. $K_i \cap A_1 = \emptyset$ *or* $K_i \cap A_2 = \emptyset$, *for* $i = 1, 2$.

Proof. Without loss of generality, suppose to the contrary that there exist $a \in A_1$ and $b \in A_2$ such that $a, b \in K_1$. Being K_1 proper to K_2, there exist $c, d \in K_2$ (possibly non-distinct) such that $ad, bc \notin E$. We first show that $c, d \notin A_1 \cup A_2$. Note that $d \notin A_1$ and $c \notin A_2$. Now suppose that $d \in A_2$; it follows that $d \neq c$. Since c is adjacent to d and not adjacent to b, and $\{A_1, A_2\}$ is a homogenous pair, it follows that $c \in A_1$. But then a, b, c, d induce a C_4 on $G[A_1 \cup A_2]$, and therefore neither a is universal to c nor c is universal to a, which is a contradiction to Lemma 12. We get an analogous contradiction if we assume that $c \in A_1$.

So $c, d \notin A_1 \cup A_2$; being $ad, bc \notin E$ and $\{A_1, A_2\}$ a homogeneous pair, c is anti-complete to A_2 and d is anti-complete to A_1. Since K_2 is a clique, it follows that $K_2 \cap (A_1 \cup A_2) = \emptyset$. Since $A_1 \cup A_2$ is not a clique, there exist $a' \in A_1$, $b' \in A_2$ such that $a'b' \notin E$. Note that $da' \notin E$ and that $a' \notin K_2$. We now show that $a' \notin K_1$. For, suppose the contrary; then $b' \neq b$ and $b' \notin K_1$, and so b' is proper to K_1 and therefore belongs to K_2, which is a contradiction, since we already argued that $K_2 \cap (A_1 \cup A_2) = \emptyset$.

Hence $a' \notin K_1 \cup K_2$. Since $\{K_1, K_2\}$ is a proper pair, there exists a vertex $e \in K_2$ such that $ea \in E$. Since $K_2 \cap (A_1 \cup A_2) = \emptyset$ and $\{A_1, A_2\}$ is a homogeneous pair, it follows that $ea' \in E$. On the other hand, we observed that $da' \notin E$. But then a' is proper to K_2, contradicting $a' \notin K_1$. (*End of the claim.*)

From the claim, we may assume without loss of generality that $K_1 \cap A_1 = \emptyset$. In this case, the statement follows if $K_2 \cap A_2 = \emptyset$, so suppose that there exists $v_2 \in K_2 \cap A_2$. It again follows from the previous claim that $K_2 \cap A_1 \neq \emptyset$; hence the statement follows if $K_1 \cap A_2 = \emptyset$. So suppose that there exists $v_1 \in K_1 \cap A_2$; since $\{K_1, K_2\}$ is a proper pair, it follows that $v_1, v_2 \in A_2$ are not universal to each other, a contradiction to Lemma 12. □

3. An algorithm for removing proper and homogeneous pairs

We now define an operation of *reduction* that is crucial for the paper. This operation essentially replaces a PH pair of cliques with a C_4^{free}H pair of cliques. The latter pair will be defined through a suitable graph that we call, for shortness, a *non-proper 2-clique*.

Definition 14. *A* non-proper 2-clique $H_{\{A_1,A_2\}}$ *is a graph with a* C_4^{free} *pair of cliques* $\{A_1, A_2\}$, *such that* $V(H_{\{A_1,A_2\}}) = A_1 \cup A_2$.

Definition 15. *Let* G *be a graph with a PH pair of cliques* $\{K_1, K_2\}$. *Also let* $H_{\{A_1,A_2\}}$ *be a non-proper 2-clique graph vertex-disjoint from* G. *The* PH reduction *of* G *with respect to* $(K_1, K_2, H_{\{A_1,A_2\}})$ *returns a new graph* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *defined as follows:*

- $V(G|_{K_1,K_2,H_{\{A_1,A_2\}}}) = (V(G) \setminus (K_1 \cup K_2)) \cup (A_1 \cup A_2)$;
- *Let* x, y *be vertices of* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$. *The edge* $xy \in E(G|_{K_1,K_2,H_{\{A_1,A_2\}}})$ *if and only if one of the following holds:*
  - $xy \in E(G)$ *with* $x, y \notin K_1 \cup K_2$;
  - $xy \in E(H_{\{A_1,A_2\}})$ *with* $x, y \in A_1 \cup A_2$;
  - $y \in A_1$, $x \notin K_1 \cup K_2$ *and* x *is complete to* K_1;
  - $y \in A_2$, $x \notin K_1 \cup K_2$ *and* x *is complete to* K_2.

We skip the trivial proof of the following lemma.

Lemma 16. *The graph* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *is such that the following properties hold:*

- $\{A_1, A_2\}$ *is a* C_4^{free}*H pair of cliques;*
- *if* $x, y \in A_1$ *(resp.* $x, y \in A_2$*), then* x *is universal to* y *or* y *is universal to* x*;*
- *if* $|K_1| \geq |A_1|$ *and* $|K_2| \geq |A_2|$, *then the graph* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *can be built in time* $O(|V(G)|^2)$ *and* $|V(G|_{K_1,K_2,H_{\{A_1,A_2\}}})| \leq |V(G)|$.

The following crucial lemma shows that all the PH pairs of $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ are "inherited" by the input graph G.

Lemma 17. *Let* $\{w_1, w_2\}$ *be a pair of adjacent vertices of* $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ *with a PH-embedding. Then:*

1. w_1 *and* w_2 *do not both belong to* $A_1 \cup A_2$*;*
2. *if* $w_1, w_2 \notin A_1 \cup A_2$, *then* $\{w_1, w_2\}$ *also admits a PH-embedding in* G*;*
3. *if* $w_1 \in A_1$ *(resp.* $w_1 \in A_2$*) and* $w_2 \notin A_1 \cup A_2$, *then, for each* $a \in K_1$ *(resp.* $a \in K_2$*),* $\{a, w_2\}$ *admits a PH-embedding in* G.

Proof. Throughout the proof, when referring to vertices of $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$, we call *artificial* the vertices of $A_1 \cup A_2$, and *non-artificial* the others. Moreover, we let $G' = G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ and let $\{K'_1, K'_2\}$ be a PH-embedding for $\{w_1, w_2\}$ in G'.

It follows from Lemma 16 that $\{A_1, A_2\}$ is a C_4^{free}H pair of cliques of G'. Therefore it follows from Lemma 13 that $K'_1 \cap A_2 = K'_2 \cap A_1 = \emptyset$ or $K'_1 \cap A_1 = K'_2 \cap A_2 = \emptyset$. Now suppose that $w_1, w_2 \in A_1 \cup A_2$, and recall that, by definition, $w_1, w_2 \in K'_1$. It follows that either $w_1, w_2 \in A_1$, or $w_1, w_2 \in A_2$. Thus, there exist two vertices of A_1 (resp. A_2) that are non-universal to each other, contradicting Lemma 16. Therefore w_1 and w_2 do not both belong to $A_1 \cup A_2$, i.e. statement 1 holds.

W.l.o.g. in the following we assume that $K'_1 \cap A_2 = K'_2 \cap A_1 = \emptyset$. Now define the sets H_1, H_2 of vertices in G as follows: for $i = 1, 2$, if K'_i has no artificial vertices, define $H_i = K'_i$; otherwise $H_i = (K'_i \cap V(G)) \cup K_i$. Note that this implies that $H_1 \cap K_2 = H_2 \cap K_1 = \emptyset$ and that H_1 and H_2 are cliques.

Claim 2. *Let* $u, v \in K'_1$ *(respectively* K'_2*) be two non-artificial vertices of* G' *such that* u *is non-universal to* v *in* G'. *Then* $u, v \in H_1$ *(respectively* H_2*) and* u *is non-universal to* v *in* G.

Proof. We prove the statement for $u, v \in K'_1$. Since u, v are non-artificial, $u, v \in H_1$ by definition. By hypothesis, there exists $z \in K'_2$ s.t. $uz \notin E(G')$, $vz \in E(G')$. If z is non-artificial, $z \in H_2$ by definition, thus u is non-universal to v in G. Suppose now z is artificial, then $z \in A_2$, since $K'_2 \cap A_1 = \emptyset$. Then by construction v is complete and u anticomplete to K_2 in G, thus u is non-universal to v in G. (*End of the claim.*)

Claim 3. *Let* $u, v \in K'_1$ *(respectively* K'_2*), and suppose* u *is artificial and* v *is not. Then* $\{v\} \cup K_1 \subseteq H_1$ *(resp.* $\{v\} \cup K_2 \subseteq H_2$*). Furthermore:*

1. *If* u *is non-universal to* v, *then* a *is non-universal to* v *for each* $a \in K_1$ *(respectively* K_2*).*
2. *If* v *is non-universal to* u, *then* v *is non-universal to* a, *for each* $a \in K_1$ *(resp.* K_2*).*

Proof. We prove the statement for $u, v \in K'_1$. We are assuming that $K'_1 \cap A_2 = \emptyset$, hence $u \in A_1$. So by definition, $\{v\} \cup K_1 \subseteq H_1$. Suppose u is non-universal to v: there exists $z \in K'_2$ s.t. $uz \notin E(G')$, $vz \in E(G')$. If z is an artificial vertex, then $z \in A_2$, which implies that v is complete to K_2, while each vertex $a \in K_1$ is proper to K_2. If z is non-artificial, then by construction z is anticomplete to K_1 while $vz \in E(G)$. This shows 1. Now suppose that v is non-universal to u, i.e. there exists $z \in K'_2$ such that $uz \in E(G'), vz \notin E(G')$. If z is an artificial vertex, then $K_2 \subseteq H_2$ and v is anticomplete to K_2; since each vertex $a \in K_1$ is proper to K_2, v is non-universal to a. If z is non-artificial, then z is complete to K_1 in G, while $zv \notin E(G)$; thus, v is non-universal to $a \in K_1$.(*End of the claim.*)

Claim 4. *$\{H_1, H_2\}$ is a PH pair of cliques in G.*

Proof. We already observed that H_1 and H_2 are cliques, and it is straightforward to see that $\{H_1, H_2\}$ is a homogeneous pair. So we conclude the proof by showing that H_1 is proper to H_2 (the other case following by symmetry).

We need to show that each vertex $x \in H_1$ has at least one neighbor and at least one non-neighbor in H_2. Recall that $x \notin K_2$. Suppose first that $x \in K_1$; then by construction $K_1 \subseteq H_1$ and K'_1 has at least one artificial vertex, say a. Since $\{K'_1, K'_2\}$ is a proper pair, it follows from Lemma 6 that there exist a vertex $t_1 \in K'_1$ to which a is non-universal, and a vertex $t_2 \in K'_1$ which is non-universal to a. If t_1 or t_2 is artificial, then K'_2 intersects A_2 (recall that $a, t_1, t_2 \in A_1$ have the same neighborhood outside K'_2) and consequently, by construction, $K_2 \subseteq H_2$; then the statement follows since $\{K_1, K_2\}$ is a proper pair of cliques. Conversely, if both t_1 and t_2 are non-artificial, then, using Claim 3, we conclude that in G x is non-universal to t_1 and that t_2 is non-universal to x, and therefore x has at least one neighbor and at least one non-neighbor in H_2.

Suppose now $x \notin K_1$: then, x is a non-artificial vertex of K'_1, and since $\{K'_1, K'_2\}$ is proper, it follows again from Lemma 6 that there exist a vertex $t_1 \in K'_1$ to which x is non-universal, and a vertex $t_2 \in K'_1$ which is non-universal to x. If both t_1 and t_2 are non-artificial, then also in G we have that x is non-universal to t_1 and t_2 is non-universal to x. If t_1 or t_2 is artificial, then thanks to Claim 3, we may suitably replace t_1 or t_2 with vertices from K_1 as to get the same conclusion. (*End of the claim.*)

We conclude the proof of the lemma: part 2 holds by Claims 2 and 4, while part 3 holds by Claims 3 and 4. □

As we show in the following, if we iterate the reduction of Definition 15, we end up, in at most $|E(G)|$ steps, with a graph without PH pairs of cliques. We first need a definition and a simple lemma, going along the same lines of Definition 15 and Lemma 17. For a graph G, we denote by $\binom{V(G)}{2}$ the set of unordered pairs of vertices of $V(G)$.

Definition 18. *Let G and $G' := G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ be as in Definition 15, and let $S \subseteq \binom{V(G)}{2}$. The set $S|_{K_1,K_2,H_{\{A_1,A_2\}}} \subseteq \binom{V(G')}{2}$ is the set of pairs $\{x, y\}$ such that one of the following hold:*

- *$\{x, y\} \in S$ and $x, y \notin A_1 \cup A_2$;*
- *$x \in A_1$, $y \notin A_1 \cup A_2$ and $\{\{a, y\} \mid a \in K_1\} \subseteq S$;*
- *$y \in A_2$, $x \notin A_1 \cup A_2$ such that $\{\{x, a\} \mid a \in K_2\} \subseteq S$.*

Corollary 19. *Let G, $G' := G|_{K_1,K_2,H_{\{A_1,A_2\}}}$, S and $S' := S|_{K_1,K_2,H_{\{A_1,A_2\}}}$ be as in Definition 15 and Definition 18.*

(i) If S is a superset of $PH(G)$, then S' is a superset of $PH(G')$.

(ii) If $|K_1| \geq |A_1|$ and $|K_2| \geq |A_2|$, then $|S'| < |S|$ and S' can be built from S in time $O(|V(G)|^2)$.

Proof. (*i*) Pick any pair $\{w_1, w_2\}$ of vertices of G' which admit a PH-embedding in G': by part (1) of Lemma 17, they cannot both belong to $A_1 \cup A_2$. Suppose that $w_1, w_2 \notin A_1 \cup A_2$. Then, by part (2) of Lemma 17, $\{w_1, w_2\}$ also have a PH-embedding in G and thus $\{w_1, w_2\} \in S$. Then, by construction, $\{w_1, w_2\} \in S'$. Now, suppose that exactly one of them belongs to $A_1 \cup A_2$, w.l.o.g. w_1, and let first $w_1 \in A_1$; then by part (3) of Lemma 17, for each $a \in K_1$, $\{a, w_2\}$ is a pair of vertices with a PH-embedding in G, i.e. $\{\{a, w_2\}, a \in K_1\} \subseteq PH(G) \subseteq S$. Then, by construction, $\{w_1, w_2\} \in S'$. A similar argument works for $w_1 \in A_2$. (*ii*) The statements holds easily by construction. □

We are now ready to give our algorithm, see Algorithm 2 in the following. Note that it is fully determined, but for the choice of the non-proper 2-clique graph $H_{\{A^i_1,A^i_2\}}$ to be used in each iteration i. In fact, the definition of $H_{\{A^i_1,A^i_2\}}$ will in general depend on G^i, K^i_1 and K^i_2: this will be discussed in the next section. Given our previous arguments, it is easy to conclude that Theorem 2 correctly predicts the output and the time complexity of Algorithm 2: we skip details.

Let us remark here that in Algorithm 2 we start with a set $S^0 = E(G)$, since we assumed no prior knowledge is available on the pair of vertices of G that are candidate to have a PH-embedding. For specific graphs we may have a better knowledge of those, and consequently start from a set S^0 smaller in size. This may lead to asymptotically faster implementations of Algorithm 2.

Algorithm 2 Eliminating all proper and homogeneous pairs of cliques

```
Require: A graph $G$.
Ensure: A graph $G^q$, without PH pairs of cliques, that is obtained from $G$ by successive PH reductions.
$i := 0$; $G^0 := G$; $S^0 := E(G)$;
while $S^i$ is non-empty do
    pick a pair $\{u, v\} \in S^i$;
    using Algorithm 1 check whether the pair $\{u, v\} \in S^i$ has a PH-embedding in $G^i$;
    if $u, v$ have a PH-embedding $\{K_1^i, K_2^i\}$ then
        let $H_{\{A_1^i, A_2^i\}}$ be a non-proper 2-clique graph vertex-disjoint from $V(G^0) \cup V(G^1) \cup \ldots \cup V(G^i)$ and such that $|K_1^i| \geq |A_1^i|$ and $|K_2^i| \geq |A_2^i|$;
        $G^{i+1} := G^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$ (see Definition 15);
        $S^{i+1} := S^i|_{K_1^i, K_2^i, H_{\{A_1^i, A_2^i\}}}$ (see Definition 18);
        $i := i + 1$;
    else
        remove the pair $\{u, v\}$ from $S^i$;
    end if
end while
$q := i$.
return $G^q$.
```

4. Preserving some graph invariant or property

In this section, we show that suitable PH reductions preserve graph invariants, such as chromatic number, stability number, and clique number, or graph properties, such as perfection, or the property of a graph of being fuzzy circular interval. Most of these reductions were in fact proposed in the literature in specific contexts, but they can actually be embedded in the unifying setting of PH reductions.

In some cases [7, 8, 10, 13] the reductions that were used have the following form: take a PH pair of cliques $\{K_1, K_2\}$ and remove some suitable set of edges between vertices of K_1 and vertices of K_2 so that, in particular, in the resulting graph, no C_4 is contained in the subgraph induced by $K_1 \cup K_2$. In another case [12] the reduction has the following form: take a PH pair of cliques $\{K_1, K_2\}$ and add all possible edges between vertices of K_1 and vertices of K_2 but one. It is easy to show that all those types of reductions can be interpreted in terms of our PH reduction, so we skip such details when presenting them. Therefore, they can be embedded into the iterative framework of Algorithm 2, and one may rely on the complexity bound given by Theorem 2.

We begin with a reduction introduced by King and Reed [8, 10] for removing edges in a PH pair of cliques while preserving the chromatic number. Recall that $\chi(G)$ denotes the chromatic, $\chi_f(G)$ the fractional, and $\omega(G)$ the clique number of a graph G.

Lemma 20. *[8] Let G be a graph and suppose that we are given a PH pair of cliques $\{K_1, K_2\}$ of G. Also, let X be a maximum clique in $G[K_1 \cup K_2]$, and let G' be the graph obtained from G by removing each edge $uv \in E(G)$ such that: $u \in K_1$; $v \in K_2$; $\{u, v\} \not\subseteq X$. Then:*

- *(i) G' can be built in time $O(|V(G)|^{\frac{5}{2}})$ (from the knowledge of G, K_1 and K_2);*
- *(ii) $\chi(G) = \chi(G')$, $\chi_f(G) = \chi_f(G')$ and each k-coloring of G' can be extended into a k-coloring of G of in time $O(|V(G)|^{\frac{5}{2}})$.*
- *(iii) $\omega(G) = \omega(G')$, and each clique of G' is also a clique of G.*
- *(iv) If G is claw-free (resp. quasi-line; perfect), then G' is claw-free (resp. quasi-line; perfect).*

(One should mention that Lemma 20 can be extended to the case where $\{K_1, K_2\}$ is a *nonskeletal* and homogeneous pair of cliques [8]. Also, Andrew King [9] pointed us that this lemma is non-trivially implied by some proofs in [2]. In that paper, Chudnovsky and Ovetsky introduce another reduction for PH pairs of cliques, which is quite similar to the one above. This reduction preserves quasi-liness, while not increasing the clique number of G. It is a simple exercise to show that the reduction in [2] can be interpreted in terms of our PH reduction. Finally, we mention that proposition (*iii*) of Lemma 20 is not stated in [8], but it is almost straightforward.)

By embedding the reduction above in the iterative framework of Algorithm 2, we can reduce the problem of computing the chromatic (resp. clique) number on a given graph G to the same problem on a graph G' without PH pairs of cliques.

Corollary 21. *From a graph G one can obtain in time $O(|V(G)|^{\frac{5}{2}}|E(G)|)$ a graph G' without PH pairs of cliques such that $\chi(G) = \chi(G')$ and $\omega(G) = \omega(G')$. One can also derive an optimal coloring of G from an optimal coloring in G' in time $O(|V(G)|^{\frac{5}{2}}|E(G)|)$, while a maximum clique in G' is also a maximum clique in G.*

As argued by Li and Zang [11], the maximum weighted clique problem in the complement of a bipartite graph can be reduced to maximum flow, and hence solved in time $O(n^3)$. By building on the latter fact (and slightly increasing the complexity), Corollary 21 can be extended to the computation of a graph G' without PH cliques that preserves the maximum weighted clique and its value.

Consider now the maximum weighted stable set problem. Oriolo, Pietropaoli, and Stauffer [12] provide a reduction that preserves the value of a maximum weighted stable set. (We refer to [12] for more details and for the precise definition of the reduction, which is actually stated for the more general class of *semi-homogeneous* pairs of cliques.) By embedding their reduction in Algorithm 2, we obtain the following lemma:

Corollary 22. *Let $G(V, E)$ be a graph with a weight function $w : V \mapsto \mathbb{R}$ defined on its vertices. In time $O(|V(G)|^2|E(G)|)$ one can build a graph G' without PH pairs of cliques such that a maximum weighted stable set of G' is also a maximum weighted stable set of G.*

Interestingly, if we now move from the maximum weighted stable set problem to the stable set polytope $STAB(G)$ of a graph G, we can also embed a result in [7] in our framework. Eisenbrand et al. show – see the remark following Lemma 5 in [7] – that each facet of the stable set polytope $STAB(G)$ is also a facet of another graph G' (obtained from G by removing edges) that does not contain any PH pair of cliques. As one easily checks (cfr. the proof of Lemma 5 in [7]), also their result can be phrased in the framework of Algorithm 2.

We now move from graph invariants to graph properties. First, Oriolo, Pietropaoli, and Stauffer [13] show that a suitable reduction of PH pairs of cliques preserves the property of a graph of being, or not being, a fuzzy circular interval graph, and they exploit this fact in an algorithm for recognizing fuzzy circular interval graphs. Their reduction can also be embedded in our framework. In fact, Theorem 2 is already used in [13] for bounding the complexity of the recognition algorithm. Finally, every PH reduction preserves perfection, and under very general conditions it does not turn a non-perfect graph into a perfect one. We give just a sketch of the proof of the latter fact, since the arguments used are quite standard.

Lemma 23. *Let G be a perfect graph with a PH pair of cliques $\{K_1, K_2\}$. Also let $H_{\{A_1,A_2\}}$ be a non-proper 2-clique graph vertex-disjoint from G. Then the graph $G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ is perfect. The converse implication holds true if A_1 is not anticomplete to A_2.*

Proof. Recall that a graph is perfect if and only if it contains neither long odd holes, nor long odd anti-holes, *long* meaning of length at least 5 [3]. Let $\{Q_1, Q_2\}$ be a homogeneous pair of cliques in a graph G: it is easy to show that each long odd-hole (resp. each long odd anti-hole) of G takes at most one vertex from Q_1 and at most one vertex from Q_2.

Suppose first that $G' = G|_{K_1,K_2,H_{\{A_1,A_2\}}}$ is not perfect, i.e. there is an induced subgraph H' of G' that is either a long odd-hole or a long anti-hole. By building on the fact that $|V(H') \cap A_1| \leq 1$ and $|V(H') \cap A_2| \leq 1$, one can easily construct an odd-hole (resp. an odd anti-hole) of G from H', thus showing that G is not perfect as well. Let now A_1 be not anticomplete to A_2 in G'; then, one can analogously show that if G is not perfect, neither is G'. □

We conclude by pointing out that, with the exception of the reduction from Lemma 20 (since $X \subseteq K_1$ or $X \subseteq K_2$ may happen), all the reductions from the current section do not turn an imperfect graph into a perfect one.

Note. While preparing this paper, we became aware that M. Chudnovsky and A. King independently found a result similar to Theorem 2, even though they reduce all proper and homogeneous pairs of cliques at once [1].

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