Stable Manifold for the Critical Non-Linear Wave Equation: A Fourier Theory Approach

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“The endeavor to understand is the first and only basis of virtue.”
— Baruch Spinoza

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Abstract

We construct a stable manifold for the Focusing Critical non-linear Wave Equation

$$\partial_{tt}\psi - \Delta\psi - \psi^5 = 0$$

in dimension 3.
More precisely we consider linearization around some static Aubin-Talenti solution and construct a subset of the phase space containing Cauchy starting values for the radiative part of the wave such that it exsits globally in positive time. Those global solutions will have the additional property to scatter to a free wave.
Taking inspiration from the article of Krieger and Schlag written in 2005, we essentially discretize the fixed point method established in the latter.
The main tool used to obtain linear dispersive estimates for the propagation operators is the so-called Distorted Fourier representation.

Contents

1 Introduction

In this text we shall investigate the possibility to obtain global in positive time radial solutions of the non-linear critical wave equation in dimension $n = 3$.

More precisely one will linearize around some static radial solution, called an *Aubin-Talenti* solution, and prove that the radiative part of the wave exists globally in positive time and further scatters to a free wave when $t \to \infty$.

The linearization process will lead to the spectral analysis of an operator, called the linearized Hamiltonian associated to the specific Aubin-Talenti solution.
When written in dimension 1 this operator has the form of a Sturm-Liouville operator. Such operators possess common features with the standard one-dimensional Laplacian in the sense that their self-adjoint extensions may be well described using a unitary representation, called the Distorted Fourier representation, sharing many common features with the usual Fourier transform.

The goal is to exploit this representation for obtaining some control of the propagation of the wave.

Two main obstacles have to be bypassed for achieving the objective.
The first one is the presence of a resonance associated to the linearized operator which behaves roughly like a zero mode (an eigenvector with eigenvalue 0) with the main difference that it is not an L^2-function. This will lead to a carefull analysis of the different norms that can be controlled in the construction of the wave.
The second difficulty is coming from the fact that the discrete spectrum of the linearized operator exhibits a negative eiegnvalue which may lead to unstable propagation (blow-up) for the wave. One has to cleverly control the propagation in this mode for obatining a global

solution. This will bring a supplementary condition, called the *stability condition*, which will be exploited for defining a stable manifold for the wave. This simply consists of a subset of the phase space regrouping the Cauchy starting values (for the wave and its time derivative) leading to globality of the solution.

The equation in concern, called precisely the *focusing critical non-linear wave equation* in $\mathbb{R}_+ \times \mathbb{R}^3$, is given by

$$\Box\psi - \psi^5 = 0, \tag{1.1}$$

where the wave $\psi = \psi(t, x)$ is a function $\psi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{C}$.

We begin by decribing the radial static (independant of time) solutions of (1.1) called the *Aubin-Talenti* solutions. Those are radial solutions ϕ of

$$\triangle\phi - \phi^5 = 0. \tag{1.2}$$

The Aubin-Talenti solutions form a set parametrized by $\mathbb{R}_+$.
In dimension $n = 3$ one can write them in two following two equivalent forms as

$$\phi(r, a) = (3a)^{1/4}(1 + ar^2)^{-1/2}, \quad a > 0 \tag{1.3}$$

or as

$$\phi(r, \lambda) = \lambda^{1/2} W(\lambda r), \quad \lambda > 0 \tag{1.4}$$

with $W(r) = (1 + r^2/3)^{-1/2}$, where $r = \|x\|$.

As was noted at the beginning one is looking for a radial solution of (1.1) of the form $\phi(., a_\infty) + u(r, t)$ with

$$\{\phi(., a)\ ,\ a > 0\}$$

being the static Aubin-Talenti solutions given in (1.3). We call u the *radiative* part of the wave ψ. $a_\infty > 0$ is a parameter value eventually being the limit of a sequence $(a_n)_n \in \mathbb{R}_+$ of approximative values obtained in a recursive scheme needed to construct the solution.

If one considers a solution of the above type u will then satisfy the associated linearized equation given by

$$\partial_{tt} u + H(a_\infty)u = N(u, \phi_\infty), \tag{1.5}$$

where the operator $H(a_\infty)$, called *the linearized operator* relative to $\phi(., a_\infty)$, has the expression

$$H(a_\infty) = -\triangle - 5\phi_\infty^4, \tag{1.6}$$

with ϕ_∞ standing for $\phi(r, a_\infty)$ and N represents the nonlinearity given by

$$10u^2\phi_\infty^3 + 10u^3\phi_\infty^2 + 5u^4\phi_\infty + u^5. \tag{1.7}$$

The associated Cauchy problem is completed by requiring that the wave u satisfies the boundary conditions

$$(u(0,.), \partial_t u(0,.)) = (u_0, u_1), \tag{1.8}$$

with the the need for the characterization of the pair of radial functions (u_0, u_1).

As was briefly mentionned at the beginning of the introduction, one of the main difficulty, in dimension $n = 3$, when aiming for a solution existing globally in positive time and scattering to a free wave as $t \to \infty$ is the occurrence of a resonance mode appearing as part of the sinus evolution of the wave and not decaying in time. This could clearly prevent the scattering to free wave because the latter has dispersive time decay in $O(< t >^{-1})$.

In [9] they solved for the latter difficulty by allowing the static part of the solution $\phi(., a)$ to depend on t modulating on the parameter a. In other words, they let the positive parameter a depend on t and consequently consider solutions of the form $\phi(., a(t)) + u(r, t)$. This brought the possibility, by finding some suitable conditions on $a(t)$, to remove the resonance term.

One of the main issues with this way of proceeding was that they could not linearize around the parameter $a(t)$ which could have required control of linearized operators depending on $t > 0$.
They instead postulate the function $a(t)$ to converge to some limit value a_∞, as $t \to \infty$ and therefore linearized around this postulated value bringing inevitably some additional terms to control in the nonlinearity.

They ended up with two mixed equations, one being a modified (because of the vanishing of resonance using $a(t)$) version of the solution of the linearized equation as in (1.5) and a second one giving condition on $a(t)$ for getting rid of the resonance giving then the possibility to obtain dsipersive decay for u.

They proved the existence of solutions $u = u(t, r)$ and $a = a(t)$ to those two mixed equations by running a fixed point argument in some complete metric space. This metric was constructed such that the solution had dispersive free positive time decay eventually scattering to a free wave when $t \to \infty$.

Beside they obtained a codimension one stable manifold of Cauchy values for such solutions by controlling the evolution of the wave in the unstable mode using the above mentionned *stability condition*.

We shall approach this problem with a slightly different point of view essentially trying to avoid the fixed point argument implying to deal with some complicated topology leading to tedious calculations.
Instead we use a sort of discretization of this fixed point argument mainly using the linear dispersive estimates for the propagation operators relative to the wave obtained via Fourier transform methods. This shall enable one to find the norms to be controlled recursively for ending up with a sequence of approximated solutions having the dispersive free time decay.

We also have to modulate in the parameter $a > 0$ for eliminating at each stage of the iterative construction the resonance terms preventing us to get the free dispersive behaviour of the wave. This whole procedure will then produce two sequences, one for the modulation of parameter $(a_n)_{n\geq 0}$ and one consisting of wave approximations $(u_n)_{n\geq 0}$.
Showing appropriate convergence for those two sequences to some u and a_∞ will permit to conclude to the existence of a global solution of (1.5) with the required properties.

The form for the stable manifold one obtains will also be slightly different from the one constructed in [9].

The main improvement compared to the result obtained in [9], aside from the substancial technical simplification of the proof, will be that the starting Cauchy values given in (1.8) will not have to satisfy the compact support assumption.

Prior to give the statement of the main theorem to be proved in this text we give some details concerning the linearized operator H_a for arbitrary $a > 0$ given in the form

$$H(a) = -\triangle - 5\phi(.,a)^4. \tag{1.9}$$

Basic spectral analysis for the self-adjoint operator H_a on $L^2(\mathbb{R}^3)$, essentially reducing matters to dimension 1 taking advantage of the radiality of the functions under consideration, gives its spectrum as $\sigma(H_a) = \{-k_a^2\} \cup [0,\infty)$ with k_a the associated square root of the absolute value of the negative eigenvalue.
$-k_a^2$ is not degenerated and the associated normalized eigenvector will be denoted by g_a.

Relating the potential $V(a) := -5\phi(.,a)^4$ for an arbitrary $a > 0$ to $V(1)$ one obtains the following scaling relations

$$\begin{aligned} k_a &= \sqrt{a}k_1 \\ g_a &\approx e^{-k_a r}, \end{aligned} \tag{1.10}$$

the second line of (1.10) being just Agmon form for the ground state.

The main objective of this text is to prove the following

Theorem 1.1. *Fix* $0 < \delta < 1$ *and let*

$$\begin{aligned} \mathscr{B}_{1,\delta} &:= \{f \in L^2(\mathbb{R}^3) \,:\, \| < r > f\|_{H^{5/2(+)}(\mathbb{R}^3)} < \delta\} \\ \mathscr{B}_{2,\delta} &:= \{g \in L^2 \,:\, \| < r > g\|_{H^{3/2(+)}(\mathbb{R}^3)} < \delta\}. \end{aligned} \tag{1.11}$$

Define then

$$\Sigma := \{(f_0, u_1) \in \mathscr{B}_{1,\delta} \times \mathscr{B}_{2,\delta} \,:\, < g_1, k_1 f_0 + u_1 >= 0\}. \tag{1.12}$$

Then there exists a Lipschitz function $h : \Sigma \to \mathbb{C}$ *such that for every pair* $(f_0, u_1) \in \Sigma$ *one can find a positive real number* $a_\infty(f_0, u_1) \in (1-\delta, 1+\delta)$ *such that the Cauchy problem*

$$\begin{cases} \Box\psi - \psi^5 = 0 \\ \psi(0,.) = \phi(.,1) + f_0 + h(f_0, u_1)g_1 \,,\ \partial_t\psi(0,.) = u_1 \end{cases} \tag{1.13}$$

has a unique radial global in positive time solution under the form
$\phi(., a_\infty) + u(r,t)$ *whith the radiative part* u *dispersing like a free wave and scattering to a free wave in the phase space* $\dot{H}^1 \times L^2$ *when* $t \to \infty$.

Before starting with the main text we shall briefly examine the content of the following chapters.

In chapter 2 one recalls some of the theory of the self-adjoint extensions for the symmetric operators acting on a Hilbert space. One is mainly interested in the method of boundary triplets which will be used in Chapter 3 when dealing with Sturm-Liouville operators.
For the sake of completeness and to gain a better understanding of the subject one develops the basics of the von Neumann adhoc theory and show the relationship between both approaches.

In chapter 3 one concentrates on the case of Sturm-Liouville operators which are Schrödinger operators in dimension 1. The goal is to prove a spectral theorem showing that, under certain circumstances, they are well-described using a unitary representation called the *Distorted Fourier* representation.
The 1-dimensional version of the linearized operator given in (1.9) is such a Sturm-Liouville operator.

In chapter 4 one describes the Fourier basis associated to (1.9). This is the most technical part of the thesis and is crucial for the entire subsequent development mainly because it gives one the possibility to perform concrete calculations.

In chapter 5 the so-called linear dispersive estimates for the propagation operators related to the linearized wave equation (1.5) are established. They represent the main tool to be used in the upcoming proof of Theorem 1.1 given in chapter 6.

It is to be mentionned that the following references, even if not cited in the main text, were consulted many times during the thesis work.
One is making reference to [3], [8], [12], [13] and [16]

2 Self-adjoint Extensions of Symmetric Operators

In this chapter we shall recall some facts from the theory of self-adjoint extensions for general densely defined symmetric operators. We shall not go for too much details just enough for the reader to understand the main ideas.
We start with discussing the von Neumann extension of a symmetric operator. This is based on a the Cayley transform which permits to replace the difficult problem of finding symmetric extensions of an operator T by the easier one of finding isometric extensions of an associated isometric operator V_T called the Cayley transform of T.
After having set up this stage, we shall, in a second section, develop the theory of boundary triplets which is a very powerful tool for parametrizing all self-adjoint extensions of a densely defined symmetric operator.
Finally we shall quickly see how the two methods are linked and give some examples for the easiest differential operators in dimension one.

The totality of the results presented in this chapter is already known and can be found in standard texts on unbounded linear operators in Hilbert spaces such as [15], [12]/ [14] or [18].

von Neumann Extensions

The Cayley Transform

We are working here with symmetric operators defined on an Hilbert space H. The idea of the Cayley transform is to perform on operators a transformation similar to the Mobius transform $t \to (t-\lambda)(t-\overline{\lambda})^{-1}$, with $\lambda \in \mathbb{C}, Im(\lambda) > 0$, mapping the real axis onto the set $\mathbb{T} \setminus \{1\}$ with $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. This transformation, when extended to $\mathbb{C}$, maps the upper complex plane onto the inside of the unit circle and the lower one onto the outside. We shall see that an analogeous map can be applied to densely defined symmetric operators T giving us all the isometric operators V such that $R(I-V)$ is dense (where R stands for the range of the operator). This last property, indicating among others that V is not the identity, is to be put in parallel to the fact that the image of the Cayley transform does note include 1.

Before proceeding with the main results of this section one defines the *defect numbers* of an operator.

Let T be a densely defined symmetric operator defined on Hilbert space H.

The defect numbers indicates how far an injective operator of the form $T - \lambda I$, having a bounded inverse defined on its range, is prevented from being invertible.
Recall that an (injective) operator $A : D(A) \to H$ is defined to be invertible if it exists a bounded B whith $D(B) = H$ and $AB = Id_H, BA = Id_{D(A)}$.
B (which is clearly uniquely defined) is called the inverse for A and written A^{-1}.

Define the set

$$\pi(T) = \{\lambda \in \mathbb{C} \mid \|T - \lambda I(x)\| \geq c_\lambda \|x\|, \text{ with } c_\lambda > 0 \text{ and all } x \in D(T)\},$$

where $D(T)$ stands for the domain of T and $c_\lambda > 0$ is some constant independant of x.
For $\lambda \in \pi(T)$, $T - \lambda I$ has thus a bounded inverse defined on $R(T - \lambda I)$.

Given $\lambda \in \pi(T)$, the *defect number* $d_\lambda(T)$ is defined to be $dimR(T - \lambda I)^\perp$.

It can be shown that $\pi(T)$ is open and that the defect number function d_λ is constant on each connected component of $\pi(T)$.

Moreover, if T is closed, one observes that the resolvent set of T is given by those numbers $\lambda \in \pi(T)$ for which $d_\lambda(T) = 0$.
T has to be closed for the resolvent set not to be empty.
In fact if λ is in the resolvent set then $(T - \lambda I)^{-1}$ will be closed, essentially using the closed graph theorem ($(T - \lambda I)^{-1}$ is bounded and $D((T - \lambda I)^{-1}) = H$), and then T will also be closed using the fact that $T - \lambda I$ is.

From now on in this section we therefore consider that T is closed.
Note that from the adjoint point of view this restriction has essentially no impact because is one considers an arbitrary operator S one has that $S^\star = \overline{S}^\star$, where $\overline{S}$ denote the closure of S.

We shall now see what happens with those defect numbers in the case of an isometric operator

V.
For $x \in D(V)$ one writes

$$\|(V-\mu I)(x)\| \geq |\|V(x)\| - |\mu|\|x\|| \geq |(1-|\mu|)|\|x\|$$

concluding that $\mathbb{C} \setminus \mathbb{T} \subset \pi(V)$. We therefore note $d^e(V)$ respectively $d^i(V)$ for the (constant) defect number of V outside respectively inside of $\mathbb{T}$.
One easily shows that

$$d^i(V) = dimR(V)^{\perp} \tag{2.1}$$

and that

$$d^e(V) = dimD(V)^{\perp}. \tag{2.2}$$

One also gets the following result needed in the proof of some fundamental properties of the upcoming Cayley Transform.

Lemma 2.1. *If V is isometric on H and $R(I-V)$ is dense, then $N(I-V) = \{0\}$ where N stands for the kernel.*

Proof. Letting $v \in D(V)$ and $x \in N(I-V)$ one gets, using the fact that V preserves the scalar product (by the use of the parallelogram law),

$$<(I-V)v, x> = <v, x> - <v, x> = 0.$$

One concludes, using the density of $R(I-V)$, that $x = 0$. □

Let T be a densely defined symmetric operator amd consider the invertible operator $T - \overline{\lambda} I$ for $Im(\lambda) > 0$. One then defines the *Cayley Transform* of T by

$$V_T = (T-\lambda I)(T-\overline{\lambda} I)^{-1}. \tag{2.3}$$

We shall list the properties of V_T relevant for subsequent developments.

Proposition 2.2.
1. *V_T is an isometric operator defined on $R(T-\overline{\lambda} I)$ with $R(V_T) = R(T-\lambda I)$*
2. *$R(I-V_T) = D(T)$*
3. *$T = (\lambda I - \overline{\lambda} V_T)(I-V_T)^{-1}$*
4. *V_T is closed*
5. *if S is another symmetric operator such that $T \subseteq S$, then $V_T \subseteq V_S$*

6. *$d^i(V_T) = d_-(T)$ and $d^e(V_T) = d_+(T)$ where $d_{+/-}(T)$ are the defect numbers of the symmetric operator T for the lower respectively the upper half-plane.*

Proof. The technical details for the proof of (1), (3) and (5) can be found in [15] (Thm F.6, p. 414).

The fact that $D(V_T) = R(T - \overline{\lambda}I)$ and $R(V_T) = R(T - \lambda I)$ is direct from the definition of V_T.

For (2) one writes, for $x \in D(T)$, using the definition of V_T that

$$V_T(T - \overline{\lambda}I)(x) = (T - \lambda I)(x).$$

Therefore

$$(I - V_T)(T - \overline{\lambda}I)(x) = (\lambda - \overline{\lambda})x.$$

One concludes by observing that $\lambda \neq \overline{\lambda}$ because $Im(\lambda) > 0$.

V_T being isometric and $R(I - V_T)$ being dense (by (2) and $D(T)$ dense) one can use the lemma 2.1 to conclude that $I - V_T$ has an inverse. The right-hand side in (3) is therefore well-defined.

Concerning (4) one uses the characterization for the domain of V_T found in (1) for concluding that V_T is closed essentially taking into account that it is an isometry and that $D(V_T) = R(T-\overline{\lambda}I)$ is closed due to the closedness of T and the fact that $\overline{\lambda} \in \pi(T)$ (see [14] (Thm. VIII.3, p.256) for a proof of the last assertion about the closedness of $R(T - \overline{\lambda}I)$).

(6) is trivial to prove using essentially (2.1) and (2.2). □

Proceeding in the reverse direction one starts with an isometric V such that $R(I - V)$ is dense. By lemma 2.1 one is able to define

$$T_V = (\lambda I - \overline{\lambda}V)(I - V)^{-1}. \tag{2.4}$$

and calls T_V the *inverse Cayley transform* of V.

By Proposition 2.2 (3) one has $T_{V_T} = T$.
It is not hard to prove (See...Schmudg) that the inverse Cayley transform is the inverse of the Cayley transform considering the latter as a map between the set of densely defined symmetric operators and the closed isometries V such that $R(I - V)$ is dense.

An important consequence of the discussion so far is

Corollary 2.3. *A densely defined symmetric operator T is self-adjoint iff its Cayley transform V_T is unitary*

Proof. One can show (see Schmudg or Reed/Simon) that a symmetric operator T is self-adjoint iff $R(T-\lambda I) = R(T-\overline{\lambda} I) = H$ for some $Im(\lambda) > 0$. The conclusion is then clear because the former range is $R(V_T)$ and the latter is $D(V_T)$. □

This corollary gives us a strong clue for deciding when a symmetric extension of a symmetric operator is self-adjoint. One has then to check that the Cayley transform of the former is unitary.
This is the main idea behind the von Neumann extension theorem presented in the next subsection.

For future use we give another property pecising when a unitary operator is the Cayley transform of a (self-adjoint) operator.

Proposition 2.4. *A unitary operator V on H is the Cayley transform of some self-adjoint operator iff $N(I-V) = \{0\}$*

Proof. One knows that V is the Cayley transform of a self-adjoint operator iff $R(I-V)$ is dense in H. One checks that unitarity of V implies that this last property is equivalent to $N(I-V) = 0$. □

von Neumann Theorem

One begins with

Proposition 2.5. *A self-adjoint operator S is an extension of a densely defined closed symmetric operator T iff it is a restriction of $T^\star$*

Proof. If S is an extension, then $T \subseteq S$ implies $S = S^\star \subseteq T^\star$. Conversely, if $S \subseteq T^\star$ then $S = S^\star \supseteq T^{\star\star} = T$, the last equality holding because T is closed. □

Thus a self-adjoint extensions S of a symmetric operator T will be completely characterized by its domain.

We get the following (the proof is adapted from [15] (Thm 13.9, p.288))

Theorem 2.6. *Let T be a densely defined symmetric closed operator and suppose that $G_+ \subseteq N(T^\star - \lambda I)$ and $G_- \subseteq N(T^\star - \overline{\lambda} I)$ are closed linear subspaces of the same dimension. Let U be an isometric mapping from G_+ onto G_-.*
Define T_U to be restriction of $T^\star$ to

$$D(T_U) = D(T) \oplus (I - U)G_+. \tag{2.5}$$

Then T_U is a closed symmetric extension of T. Moreover any closed symmetric extension of T is of this form and we get the following defect number relation $d_{+/-}(T) = d_{+/-}(T_U) + dimG_{+/-}$

Remark 2.7. The direct sum decomposition of $D(T_U)$ given in (2.5) is not a priori an orthogonal one. Nevertheless we do not make the distinction between an orthogonal and a non-orthogonal direct sum decomposition, the distinction being clear from the context.

Proof. We begin by proving that every symmetric extensions of T has to be of the form T_U with some triplet (G_+, G_-, U) as in the statement of the theorem and that every such triplet corresponds to some symmetric extension of T.

We consider the Cayley transform V_T of T.
One knows by Proposition 2.2 (5) and the fact that the Cayley transform is a one-one correspondance between the set of densely defined symmetric operators and the set of isometric closed operators (with some conditions on the range of the latters) that closed symmetric extensions of T are in one-to-one correspondance with closed isometric extensions V of V_T.
Thanks to the well-known decomposition $H = N(T^\star - \lambda I) \oplus R(T - \overline{\lambda} I)$ (which is valid here because T is symmetric and therefore $\overline{\lambda} \in \pi(T)$) one can define such a V by a domain of the form $D(V_T) \oplus G_+$ with $G_+ \subset N(T^\star - \lambda I)$ and $V(y) = V_T(y)$ if $y \in D(V_T)$ and $V(y) = U(y)$ if $y \in G_+$ with G_+ and U as in the statement of the theorem.
We finally consider the inverse Cayley transform of V and write $T_U := (\lambda I - \overline{\lambda} V)(I - V)^{-1}$ noting that it is well-defined because V is an isometric extension of V_T implying that $R(I - V)$ is dense.
For the domain of T_U one gets

$$\begin{aligned} D(T_U) = (I - V)D(V) = (I - V)(D(V_T) \oplus G_+) = \\ (I - V_T)D(V_T) \oplus (I - U)G_+ = D(T) \oplus (I - U)G_+ \end{aligned} \tag{2.6}$$

the sum remaining direct by the injectivity of $(I - V)$ (cf lemma 2.1).
Since $T \subseteq T_U$ and T_U is symmetric, we get $T_U \subseteq T_U^\star \subseteq T^\star$.
T_U is therefore of the required form.

The conclusion about the defect numbers is an easy consequence of proposition 2.2 (6) and the formulae (2.1) and (2.2) taking advantage of the fact that the direct sum decomposition of $D(V)$ is an orthogonal one. □

Remark 2.8. 1. $D(T_U)$ being a subset of $D(T^\star)$ one can recognize the decomposition of the former accordingly to the von Neumann decomposition

$$D(T^\star) = D(T) \oplus N(T^\star - \lambda I) \oplus N(T^\star - \overline{\lambda} I) \tag{2.7}$$

(see [18] (Thm 8.11. p.237)).

2. Considering the above decomposition for the domain $D(T^\star)$ one observes that a symmetric closed operator T is self-adjoint iff $N(T^\star - \lambda I) = \{0\} = N(T^\star - \overline{\lambda} I)$.
In other words its defect numbers have to be 0 for obtaining self-adjointness.

From Theorem 2.6 and Corollary 2.3 one obtains the von Neumann Theorem given in the form

Theorem 2.9 (von Neumann Extension)**.** *A densely defined closed symmetric operator T has self-adjoint extensions iff $d_+(T) = d_-(T)$ or, equivalently, $dimN(T^\star - \lambda I) = dimN(T^\star - \overline{\lambda} I)$ for $Im\lambda > 0$. All such extensions have the form given in Theorem 2.6.*

Proof. T has a self-adjoint extension S iff V_S is unitary by 2.3, meaning precisely by Corollary 2.6 that there should exist an isometry between $N(T^\star - \lambda I)$ and $N(T^\star - \overline{\lambda} I)$. This is only possible iff they have equal dimension. The rest of the theorem is an immediate consequence of 2.6. □

Remark 2.10.

Regarding Remark 2.7 (2) the existence of a self-adjoint extension S of a symmetric closed operator T occurs precisely when $d_+(S) = d_-(S) = 0$.

Theorem 2.9 says that the self-adjoint extensions of a closed symmetric operator T (if it exists!) can be parametrized by the isometries U between $N(T^\star - \lambda I)$ and $N(T^\star - \overline{\lambda} I)$.

We end this section with a simple example showing the power of this approach.

Example 2.11. Suppose $a < b$, two real numbers and let T be the linear operator on $L^2(a,b)$ defined by $Tf := -if'$ for $f \in D(T) = H^1_0(a,b)$. The latter space is just $H^1(a,b)$ with functions vanishing at a and b. One can show that the symmetric operator T is closed and has its adjoint given by the same operator form but with domain $D(T^\star) = H^1(a,b)$. Moreover it is easy to observe that

$$N(T^\star - \lambda I) = \mathbb{C} \exp i\lambda x$$

and that clearly $d_+(T) = d_-(T) = 1$. T has thus self-adjoint extensions as was shown in Theorem 2.9.
We shall apply the construction given in the proof of Theorem 2.6 to find all the extensions.
Choosing $\lambda = i$ (note that this does not change the question of extensions because the dimensions of kernels, or defect numbers, are constant on both halves of the complex plane) the isometries from N_+ to N_- (with the notation $N_\pm := N(T^\star \mp iI)$) are given by $U_w(\exp(a+b-x)) =$

$w \exp x$, parametrized thus by $w \in \mathbb{T}$. One notes that the specific form of the basis vector chosen, i.e. $\exp(a+b-x)$ and $\exp x$, is to ensure that they have the same L^2 norm.
By Theorem 2.6 one knows that $f \in D(T_{U_w})$ iff $f = f_0 + \alpha(I - U_w)\exp(a+b-x)$ with $\alpha \in \mathbb{C}$ and $f_0 \in H_0^1(a,b)$.
Having performed some additional technical computations, one concludes that the self-adjoint extensions S_z, $z \in \mathbb{T}$, of T are compactly written as $D(S_z) = \{f \in H^1(a,b) : f(b) = zf(a)\}$.

The Method of Boundary Triplets

As mentionned at the beginning of the chapter, other methods are available for finding self-adjoint extensions.
We shall present in this section another such method, based on the theory of boundary triplets, which will reveal to be really powerful and efficient when dealing with the Sturm-Liouville operators in the next chapter, operators whose deficiency spaces are harder to write than in the simple example 2.11.
We shall begin by introducing the setup of linear relations needed for using efficiently the boundary triplets associated to the adjoint of a symmetric operator.
Then we shall describe the method associated to boundary triplets. In particular, one shows how to recover the self-adjoint von Neumann extensions obtained in the preceding section using this new technique.

Linear relations

A linear relation is basically a generalization of the notion of the graph of an operator.
On some Hilbert space H one defines a *linear relation* on H being a linear subspace of $H \oplus H$.
The graph of an operator is an example of a linear relation.

One has more flexibility defining new relations from old ones than if we were dealing exclusively with operators. We shall be interested principally in the following:
let $\mathscr{T}$ be a linear relation and consider

$$\mathscr{T}^{\star} = \{(u,v) : <x,v> = <y,u> \ \forall (x,y) \in \mathscr{T}\}$$

and

$$\mathscr{T}^{-1} = \{(u,v) : (v,u) \in \mathscr{T}\}.$$

$\mathscr{T}^{\star}$ and $\mathscr{T}^{-1}$ are linear relations called respectively the self-adjoint relation associated to $\mathscr{T}$ and the inverse relation of $\mathscr{T}$.
Moreover it is almost evident that if the relation $\mathscr{T}$ is an operator then the above two associated relations define the graphs of the standard adjoint and inverse of $\mathscr{T}$ if the latter are existing.

We shall also need the following subspaces of H defined by

$$D(\mathcal{T}) = \{x : (x, y) \in \mathcal{T} \text{ for some } y \in H\},$$

$$R(\mathcal{T}) = \{y : (x, y) \in \mathcal{T} \text{ for some } x \in H\},$$

$$N(\mathcal{T}) = \{x : (x, 0) \in \mathcal{T}\},$$

$$M(\mathcal{T}) = \{y : (0, y) \in \mathcal{T}\}.$$

They are named in the obvious way except for the last one which is called the *multivalued part* of $\mathcal{T}$.
This last subspace is clearly null if $\mathcal{T}$ is the graph of a linear map.
One interprets $M(\mathcal{T})$ as a measure of how the relation $\mathcal{T}$ differs from being a graph.

Apart from the usual relations between the range of a linear operator and the kernel of its adjoint which remain valid in the case of linear relations one can prove the following

$$D(\mathcal{T})^{\perp} = M(\mathcal{T}^{\star}) \tag{2.8}$$

If an operator T is not densely defined and is considered as a linear relation, then its adjoint has a non-zero multivalued part meaning that it cannot be the graph of an operator. This is the kind of generalization one can perform when considering linear relations. We shall introduce other properties and constructions concerning linear relations along the way.

We will mostly be interested in closed relations.
Closed relations are just relations closed in the Hilbert $H \oplus H$ equipped with the usual product norm.
An interesting construction concerning the link betwenn general linear relations and the graphs of operators goes as follows.
Suppose we are given a closed linear relation $\mathcal{T}$ and define the closed relation $\mathcal{T}_m := \{(0, y) \in \mathcal{T}\} = \{0\} \oplus M(\mathcal{T})$ (m standing for multivalued). Then the orthogonal complement of $\mathcal{T}_m$ in $\mathcal{T}$ noted as $\mathcal{T}_{op} := \mathcal{T} \ominus \mathcal{T}_m$ is the graph of a closed operator.

Our next goal is to characterize self-adjoint relations.

Definition 2.12. A linear relation $\mathcal{T}$ is called *symmetric* if $\mathcal{T} \subset \mathcal{T}^{\star}$ and *self-adjoint* when we have equality

Therefore $\mathcal{T}$ is symmetric iff

$$< v, x >=< u, y > \ \forall \ (x, y), (u, v) \in \mathcal{T}.$$

Define the space $S(H)$ to be the space of all self-adjoint operators B defined on a closed subspace $H_B \subset H$. It should be noted that this means that B is defined on a dense subspace of H_B and $R(B) \subset H_B$.

We have the following proposition concerning the characterisation of self-adjoint relations

Proposition 2.13. *There is a one-to-one correspondance between operators $B \in S(H)$ and self-adjoint relations $\mathscr{B}$ on H given by*

$$\mathscr{B} = \mathscr{G}_B \oplus (\{0\} \oplus H_B^{\perp}),$$

where $\mathscr{G}_B$ is $\mathscr{B}_{op}$ and $H_B^{\perp}$ is $M(\mathscr{B})$

Proof. Given a $B \in S(H)$, it is easy to prove that the $\mathscr{B}$ defined above is self-adjoint.
Now consider a self-adjoint relation $\mathscr{B}$ and write it as $\mathscr{G}_B \oplus (\{0\} \oplus H_B^{\perp})$, where we defined H_B by $H_B := M(\mathscr{B})^{\perp}$, noting that, by closedness of $\mathscr{B}$, one gets $H_B^{\perp} = M(\mathscr{B})^{\perp\perp} = M(\mathscr{B})$.

We will show that the operator B is in $S(H)$ with domain H_B.

Using (2.8) one writes $D(B)^{\perp\perp} = D(\mathscr{B})^{\perp\perp} = M(\mathscr{B})^{\perp} = H_B$ concluding that $D(B)$ is a dense subspace in H_B. Furthermore it is almost obvious that $R(B) \subset H_B$ (by orthogonality and the closedness of H_B). Since $\mathscr{B}$ is self-adjoint, so is B. □

We saw already that there is a strong link between self-adjoint and unitary operators when discussing the Cayley transform in the preceding section.
Similarly it is possible to describe self-adjoint relations using unitary operators:

Proposition 2.14. *A relation $\mathscr{B}$ is self-adjoint iff there is a unitary operator V on H such that*

$$\mathscr{B} = \{(x, y) : (I - V)y = i(I + V)x\} \tag{2.9}$$

Moreover, the operator V is uniquely determined by $\mathscr{B}$ and called the Cayley transform of $\mathscr{B}$. Conversely, each unitary on H is the Cayley transform of some self-adjoint relation. Thus there is a one-to-one correspondance between unitaries and self-adjoint relations on H.

Proof. Suppose $\mathscr{B}$ is self-adjoint and consider the expression given by Proposition 2.13. Since the operator B is self-adjoint on H_B its Cayley transform, denoted V_B, is unitary on H_B. We extend it on H by defining it to be identity on $H_B^{\perp}$

$$Vx = (I - P_B)x + V_B P_B x,$$

where P_B is the orthogonal projection on H_B. This V is unitary on H and we easily derive the representation (2.9).

Note that V is uniquely determined by $\mathscr{B}$ essentially by construction.

Conversely, let V be an arbitrary unitary operator on H. We wish to use the inverse Cayley transform but we have to be careful because $R(I-V)$ need not be dense. One way to proceed is by reducing V.
We consider $H_B := N(I-V)^{\perp}$ (orthogonal complement of the kernel of
$I-V$) and remark that it is reducing V. Then V_{H_B} has the property that $N(I-V_{H_B}) = \{0\}$ and thus, by unitarity of V_{H_B} on H_B, V_{H_B} is the Cayley transform of a self-adjoint operator B on H_B by Proposition 2.4.
The operator B is given by the inverse Cayley transform as $B = i(I+V_{H_B})(I-V_{H_B})^{-1}$ and the associated relation $\mathscr{B}$ is then self-adjoint by Proposition 2.13. The representation (2.9) follows from first part of the proof using that $V = V_{H_B}$ on H_B and $V = I$ on $H_B^{\perp}$. □

Remark 2.15. If one considers an operator V on H one way to define reduction of V by a closed subspace $U \subset H$ is to ask $P_U D(V) \subset D(V)$, with P_U the projection on U, and that U is invariant under the action of V; in other words $V(D(V) \cap U) \subset U$.
In the context of Proposition 2.14 the operator V was defined on all of H and therefore reducing is the same as invariant.

Boundary Triplets

We are now in a position to define boundary triplets and show how one can use them for parametrizing all self-adjoint extensions of a densely defined closed symmetric operator. We shall also show how the method presented here can be used to recover the von Neumann extensions found in section 2.1.

One starts with a definition

Definition 2.16. Let T be a densely defined symmetric operator.
A *boundary triplet* for $T^\star$ is a triplet $(\mathscr{K}, \Gamma_0, \Gamma_1)$ consisting of a Hilbert space $\mathscr{K}$ and linear mapping $\Gamma_0 : D(T^\star) \to \mathscr{K}$ (same for Γ_1) such that the following hold:

1. $$\begin{gathered} \forall x, y \in D(T^\star) \\ < T^\star x, y > - < x, T^\star y > = < \Gamma_1 x, \Gamma_0 y > - < \Gamma_0 x, \Gamma_1 y > \end{gathered} \tag{2.10}$$

   with obvious scalar products on each side.

2. The mapping
   $$D(T^\star) \ni x \to (\Gamma_0 x, \Gamma_1 x) \in \mathscr{K} \oplus \mathscr{K}$$

   is surjective

Remark 2.17. 1. We shall not make the distimction between the different scalar products, this being clear from the context.

2. The basic idea behind the concept of boundary triplets is to recover what happens with differential operators when the scalar product is given by integration (of some Lebesgue functions). Thus (2.10) can be interpreted as an integration by part formula with $\Gamma_0 x$ and $\Gamma_1 y$ representing boundary values of the integrated functions. They are thus called abstract boundary values.

3. It often arises that on the right-hand side of (2.10) each member is evaluated at the same boundary.
   To remedy it, we say that $(\mathscr{K}, \Gamma_+, \Gamma_-)$ (where Γ_+ and Γ_- are linear maps as those in the Definition 2.16) is a boundary triplet iff (2.10) is replaced by

$$2i(< T^\star x, y > - < x, T^\star y >) =< \Gamma_- x, \Gamma_- y > - < \Gamma_+ x, \Gamma_+ y > .$$

   One observes that both definitions are equivalent by using the relations $\Gamma_0 = \frac{(\Gamma_+ - \Gamma_-)}{2i}$ and $\Gamma_1 = \frac{(\Gamma_+ + \Gamma_-)}{2}$.

Example 2.18. Within the same framework as in Example 2.11, an integration by parts gives

$$i[f,g]_{T^\star} = f(b)\overline{g(b)} - f(a)\overline{g(a)},$$

where we have noted the left-hand side of (2.10) as $[f,g]_{T^\star}$. We therefore have a boundary triplet with $\mathscr{K} = \mathbb{C}$, $\Gamma_+(f) = \sqrt{2}f(a)$ and $\Gamma_-(f) = \sqrt{2}f(b)$. Note that surjectivity is satisfied because $D(T^\star) = H^1(a,b)$.

The natural question of the existence of some boundary triplet for a particular densely defined symmetric operator T will be answered at the end of this subsection. It will be proved there that a boundary triplet exists iff $d_+(T) = d_-(T)$, that is exactly when self-adjoint extensions exist.
This is no coincidence because, as we shall see, if one has the existence of a boundary triplet one is then able to find and classify all self-adjoint extensions.

We shall now make the link between boundary triplets and proper extensions of the operator T.

Definition 2.19. A closed operator S is called a proper extension if $T \subseteq S \subseteq T^\star$

Note the fact that a proper extension does not need to be symmetric.

The goal of the further development is to show that self-adjoint extensions of T can be described by self-adjoint relations on $\mathscr{K}$.

Let T be a densely defined symmetric operator associated to some boundary triplet for its adjoint written as $(\mathscr{K}, \Gamma_0, \Gamma_1)$.

One introduces some additional terminology and notation.

Let $\mathscr{B}$ be some relation on $\mathscr{K}$ and consider the operator $T_{\mathscr{B}}$ given by restriction of $T^\star$ to the domain

$$D(T_{\mathscr{B}}) = \{x \in D(T^\star) : (\Gamma_0 x, \Gamma_1 x) \in \mathscr{B}\}.$$

Conversely considering a linear operator (not necessarily closed thus not being considered as a proper extension for the moment) S such that $T \subseteq S \subseteq T^\star$, we define its *boundary space* to be the relation

$$\mathscr{B}(S) = \{(\Gamma_0 x, \Gamma_1 x) : x \in D(S)\}.$$

It is clear, by the surjectivity property in Definition 2.16 (2), that $\mathscr{B}(T_{\mathscr{B}}) = \mathscr{B}$ for any linear relation $\mathscr{B}$.

One has the following crucial Lemma (whose proof is given considered its importance for the rest of the argumentation and is found in [15] (p.312))

Lemma 2.20. *Let $\mathscr{B}$ be a linear relation on $\mathscr{K}$ and let S be a linear operator on H such that $T \subseteq S \subseteq T^\star$ and $\mathscr{B}(S) = \mathscr{B}$ (noting that it implies $S \subset T_{\mathscr{B}}$).*
Then

1. $S^\star = T_{\mathscr{B}^\star}$.
2. $\overline{S} = T_{\overline{\mathscr{B}}}$
3. *S is closed iff $\mathscr{B}$ is closed*
4. $\overline{T} = T_{(0,0)}$

Proof. 1. $T \subseteq S$ implies that $S^\star \subseteq T^\star$. Thus both operators $S^\star$ and $T_{\mathscr{B}^\star}$ are restrictions of $T^\star$ meaning that one just needs to prove equality of their domains.

A vector $y \in D(T^\star)$ belongs to $D(S^\star)$ iff for all $x \in D(S)$ one has

$$< T^\star x, y >=< Sx, y >=< x, S^\star y >=< x, T^\star y > .$$

By Definition 2.16 (1) this means $< \Gamma_0 x, \Gamma_1 y >=< \Gamma_1 x, \Gamma_0 y >$ for all $x \in D(S)$. This equality is equivalent to the fact that $(\Gamma_0 y, \Gamma_1 y) \in \mathscr{B}^\star$ or equivalently $y \in D(T_{\mathscr{B}^\star})$.

2. We shall work with relations using (1)
Since $S^\star = T_{\mathscr{B}^\star}$ we have $\mathscr{B}(S^\star) = \mathscr{B}^\star = \mathscr{B}(S)^\star$.

Applying this to $S^\star$ and using that $\mathscr{B}^{\star\star} = \overline{\mathscr{B}}$ (with $\overline{\mathscr{B}}$ denoting the closure of $\mathscr{B}$), we get $\mathscr{B}(S^{\star\star}) = \overline{\mathscr{B}}$ implying that $S^{\star\star} = T_{\overline{\mathscr{B}}}$ and we are done because $S^{\star\star} = \overline{S}$.

3. The relatively non-trivial direction is to prove that if S is closed then $\mathscr{B}(S)$ too.
   But using (2) one gets
$$S \subset T_{\mathscr{B}} \subset T_{\overline{\mathscr{B}}} = S$$
   by closedness of S. Therefore $T_{\mathscr{B}} = T_{\overline{\mathscr{B}}}$ and the result follows.

4. Set $\mathscr{B} = \mathscr{K} \oplus \mathscr{K}$ and remark that $\mathscr{B}^\star = \{(0,0)\}$. Then one applies (1), observing that $T_{\mathscr{B}} = T^\star$, to obtain $\overline{T} = T^{\star\star} = (T_{\mathscr{B}})^\star = T_{\mathscr{B}^\star} = T_{(0,0)}$.

□

An immediate consequence of the last lemma is the following

Proposition 2.21. *There is a one-to-one correspondance between closed linear relations on $\mathscr{K}$ and proper extensions of T. Moreover $T_{\mathscr{B}}$ is self-adjoint iff $\mathscr{B}$ is.*

Proof. See [15] (Prop 14.7, p.313) for a complete proof. □

Example 2.22. If $(\mathscr{K}, \Gamma_0, \Gamma_1)$ is boundary triplet for $T^\star$, considering the (obvious) self-adjoint relations $\mathscr{B}_0 = \{0\} \oplus \mathscr{K}$ and $\mathscr{B}_1 = \mathscr{K} \oplus \{0\}$ one obtains two self-adjoint extensions of T written respectively $T_0 (:= T_{\mathscr{B}_0})$ and T_1.
We shall use T_0 in the proof of existence of a boundary triplet at the end of this subsection.
Moreover those specific extensions will appear again when one studies the properties of the Sturm-Liouville operators in the next chapter.

We can then parametrize all self-adjoint extensions of T.

Theorem 2.23. *Let $(\mathscr{K}, \Gamma_0, \Gamma_1)$ be a boundary triplet for $T^\star$. For any operator S the following conditions are equivalent:*

1. *S is self-adjoint extension of T*

2. *There exists an operator $B \in S(\mathscr{K})$ such that $S = T_B$, with T_B being the operator $T_{\mathscr{B}}$ with $\mathscr{B}$ the self-adjoint relation associated to the operator B as in proposition* (2.13).

3. *There is a unitary V on $\mathscr{K}$ such that $S = T^V$, where T^V is defined by the restriction of $T^\star$ to $D(T^V) = \{x \in D(T^\star) : V\Gamma_+ x = \Gamma_- x\}$*

Moreover the operators B and the unitary V are uniquely determined by S.

Proof. Apply Propositions 2.13, 2.14 and 2.21. □

One observes that Theorem 2.23 gives two ways for parametrizing the self-adjoint extensions of T.
One way is given regarding the set $S(\mathscr{K})$ of self-adjoint operators defined on closed subspaces of $\mathscr{K}$ and the other is by considering unitaries on $\mathscr{K}$.

As was shown in Proposition 2.14 those two ways are linked by the fact that the unitaries in the second parametrizing case are the Cayley transforms of the self-adjoint relations generated by the members of $S(\mathscr{K})$.

We illustrate the method by continuing the Example 2.11.

Example 2.24. Within the same framework as in Example 2.11 and using the boundary triplet given in Example 2.18 one applies Theorem 2.23 (3) and immediately recovers the results already obtained in Example 2.11 about the self-adjoint extensions of T simply because unitaries on $\mathbb{C}$ are given by multiplication by z, $z \in \mathbb{T}$.

We finish this chapter by giving the proof for the existence of a boundary triplet associated to a the adjoint of a densely defined symmetric operator T mentionned after the Example 2.18. We shall see that the proof gives us the von Neumann extensions of T given in Theorem 2.9.

The proof of the following theorem is an adaptation of two results found and proved in [15] (p.315 and Lemma 14.13 (ii), p.322)

Theorem 2.25. *There exists a boundary triplet $(\mathscr{K}, \Gamma_0, \Gamma_1)$ for $T^\star$ iff the symmetric T has equal defect numbers as $d_+(T) = d_-(T) = dim(\mathscr{K})$.*

Proof. Suppose that a boundary triplet for $T^\star$ exists.
We shall use the extension T_0 considered in Example 2.22.
If S is a self-adjoint extension of T one has the following decomposition of $D(T^\star)$ (see Schmudg...)

$$D(T^\star) = D(S) \oplus N(T^\star - zI)$$

for all $z \in \rho(S)$ (in particular this holds for every $z \in \mathbb{C} \setminus \mathbb{R}$).
Applying the latter decomposition in the case of the extension T_0 one obtains $D(T^\star) = D(T_0) \oplus N(T^\star - iI)$ and $D(T^\star) = D(T_0) \oplus N(T^\star + iI)$.
Since $N(\Gamma_0) = D(T_0)$ and using the surjectivity property of Γ_0 (Definition 2.16 (2)) one concludes that Γ_0 sends $N(T^\star - iI)$ and $N(T^\star + iI)$ bijectively onto $\mathscr{K}$.

Conversely suppose that the defect numbers are equal to the dimension of $\mathscr{K}$.
The von Neumann decomposition (2.7) says that an $x \in D(T^\star)$ can be written uniquely as a

sum $x = x_0 + x_+ + x_-$ where $x_0 \in D(\overline{T})$ and $x_\pm \in N(T^\star \mp iI)$.
Defining $Q_\pm x := x_\pm$ this is no issue to prove that $(N(T^\star + iI), 2WQ_+, 2Q_-)$, with W a unitary between $N(T^\star - iI)$ to $N(T^\star + iI)$ (existing by hypothesis), is a boundary triplet for $T^\star$. This ends the proof. □

Remark 2.26. As mentionned before Theorem 2.25 one is now able to recover the von Neumann extensions of T given in Theorem 2.9 using essentially the boundary triplet constructed in Theorem 2.25 and the parametrization of self-adjoint extensions of T by unitaries and given in Theorem 2.23 (3).
To see this consider a unitary V on $\mathscr{K} (= N(T^\star + iI))$ and define the unitary $U := -VW$ from $N(T^\star - iI)$ to $N(T^\star + iI)$ with W the unitary appearing in the proof of Theorem 2.25.
One then recovers (2.5) with $G_+ = N(T^\star - \lambda I)$ just by application of Theorem 2.23 (3).

3 Sturm-Liouville Operators

In this chapter we shall develop the theory of Sturm-Liouville operators and recall the proof of a spectral theorem for them taking advantage of the so-called Distorted Fourier representation.

In the first section we shall discuss the basics of Sturm-Liouville operators using the boundary triplets theory developed in section 2.2.
Most of the discussion, except the use of boundary triplets, has its roots in the work of H. Weyl.
We shall then explain in details the Distorted Fourier representation and give the proof of the associated spectral theorem.

An intermediary section is devoted to a slight investigation of the theory of Herglotz functions as the latter constitute an important tool in proving the spectral theorem.
We shall see that those type of functions have the desirable property that they have an integral representation w.r.t. some regular Borel measure on the line the latter being uniquely defined by the function itself.

All the results presented in this chapter are already known. We shall give the main proofs in order to keep the text a maximum self-contained.

Sturm-Liouville operators: Basics

The main concern for this chapter are second-order differential operators of the form

$$\mathscr{L} = -\frac{d^2}{dx^2} + V(x),$$

acting on $L^2(a,b)$, with $\infty \leq a < b \leq \infty$.
$V(x)$ is called the potential part of $\mathscr{L}$ and is a real-valued continuous function on (a,b).

The behaviour of V near the boundary points a, b will be of particular importance for the following developments.

One is now looking for some domain making the formal operator expression $\mathscr{L}$ into a densely defined closed symmetric operator for then being able to apply the theory developed in chapter 2.

One begins with the symmetric opreator $\mathscr{L}_0$ defined by

$$\mathscr{L}_0 f = \mathscr{L} f$$

with dense domain $D(\mathscr{L}_0) = C_0^\infty(a, b)$.

One wants to show that $\mathscr{L}_0$ is closable such that he will be in a position for considering its (symmetric) closure denoted $\mathscr{L}_{min}$.
In a second step he shall prove that the adjoint of $\mathscr{L}_{min}$ is given by the same formal operator expression $\mathscr{L}$ but with maximal domain, that is

$$D(\mathscr{L}_{max}) := \{f \in L^2(a, b) : \ f, f^{'} \in AC[\alpha, \beta] \text{ for } [\alpha, \beta] \subset (a, b), -f^{''} + Vf \in L^2(a, b)\}$$

with $AC[.,.]$ meaning the space of absolutely continuous functions on the indicated interval.

One then gets (the proof of the next Lemma can be found in [15] (p.344) and is given for the illustration as how a closure of a symmetric operator is obtained)

Lemma 3.1. *$\mathscr{L}_0$ is closable and if one denotes its (symmetric) closure by $\mathscr{L}_{min}$ then $\mathscr{L}_{min}^\star = \mathscr{L}_{max}$*

Proof. Let $\phi \in C_0^\infty(a, b)$ and $f \in D(\mathscr{L}_{max})$.
One uses integration by parts (twice) to obtain $< \mathscr{L}_0 \phi, f > = < \phi, \mathscr{L}_{max} f >$.
Therefore $\mathscr{L}_{max} \subseteq \mathscr{L}_0^\star$ and, $\mathscr{L}_0^\star$ being densely defined, $\mathscr{L}_0$ is closable.
Write its closure $\mathscr{L}_{min}$.
Using that for a densely defined operator T with adjoint $T^\star$ one has $T^\star = \overline{T}^\star$, one writes $\mathscr{L}_{min} \subseteq \mathscr{L}_{max} \subseteq \mathscr{L}_{min}^\star$ letting him to conclude that $\mathscr{L}_{min}$ is symmetric.

To prove the reverse inclusion $\mathscr{L}_{min}^\star \subseteq \mathscr{L}_{max}$, regarding the last paragraph, one is left to show that $D(\mathscr{L}_{min}^\star) \subseteq D(\mathscr{L}_{max})$.
Let $f \in D(\mathscr{L}_{min}^\star)$ and consider $g := \mathscr{L}_{min}^\star f$.
We will show that $f, f^{'}$ are AC on compact subintervals of (a, b) and that $-f^{''} + Vf = g$ in distributional sense giving us that $f \in D(\mathscr{L}_{max})$ because $g \in L^2(a, b)$.

Define h by

$$h(x) := \int_c^x \int_c^s (Vf - g)(t)\,dt\,ds$$

with $c \in (a,b)$ arbitrary. Note that h is well-defined because $Vf - g \in L^1_{loc}(a,b)$ using $f, g \in L^2(a,b)$ and V continuous on (a,b).
The goal is to show that $f'' = h''$ in distributional sense.
The proof will then be complete because, using basic distribution theory, one will be able to write $f = h + c_0 + c_1 x$ on (a,b) with c_0, c_1 constants. This imply that all the requirement for f to be in $D(\mathscr{L}_{max})$ will be satisfied regarding the form of h.

One writes for $\phi \in C_0^\infty(a,b)$

$$< f, -\overline{\phi}'' + V\overline{\phi} >=< f, \mathscr{L}_{min}\overline{\phi} >=< \mathscr{L}^\star_{min} f, \overline{\phi} >=< g, \overline{\phi} >=< Vf - h'', \overline{\phi} >$$

resulting in $< f, \overline{\phi}'' >=< h'', \overline{\phi} >$.
We used $\overline{\phi}$ for matching the scalar product notation with the distributional action of the first argument. □

For consistency with the notations used in Chapter 2 one writes $T := \mathscr{L}_{min}$ having then $T^\star = \mathscr{L}_{max}$.

Before continuing the discussion concerning the self-adjoint extensions of T we shall enumerate standard definitions and propositions about the operator expression $\mathscr{L}$.

Definition 3.2. The operator expression $\mathscr{L}$ is called *regular* at the endpoint a if V is integrable near a, meaning there exists some $c \in (a,b)$ such that $V \in L^1(a,c)$. Otherwise it is called *singular*. The same holds for the endpoint b.
Moreover an endpoint is called regular if $\mathscr{L}$ is regular at this endpoint.

Remark 3.3.

1. The former definition can be naturally extended to any point in the interval (a,b) by saying that $\mathscr{L}$ is regular at a point $c \in (a,b)$ iff $V \in L^1$ near to it, with the obvious meaning for the L^1 property. In the same way any point inside the interval is regular if $\mathscr{L}$ is regular at this point.
   With this trivial definition extension any point inside the interval (a,b) is clearly regular regarding the continuity of V.

2. If $\mathscr{L}$ is regular at the endpoint a say, that is if a $c \in (a,b)$ exists as described in the preceding definition, the precise value for c is irrelevant due to the continuity of V.

Given $\lambda \in \mathbb{C}$ and $g \in L^1_{loc}(a,b)$, we say that $f : (a,b) \to \mathbb{R}$ is a solution of $\mathscr{L}f - \lambda f = g$ if f, f' are AC on compact subinteravls of $[a,b]$ and $\mathscr{L}f - \lambda f = g$ holds a.e. in (a,b)

Proposition 3.4. *Let $\lambda \in \mathbb{C}$ and $g \in L^1_{loc}(a,b)$. Then*

1. *If $\mathscr{L}$ is regular at a (or b), then any solutions f of $\mathscr{L}f - \lambda f = g$ can be continuously extended to a (respectively b)*

2. *let $c \in [a,b]$ be regular and let $(c_1, c_2) \in \mathbb{C}^2$.*
   Then there is a unique solution of $\mathscr{L}f - \lambda f = g$ satiysfying $f(c) = c_1, f^{'}(c) = c_2$. In other words the solution set of the preceding Cauchy problem can be parametrized by $\mathbb{C}^2$.
   Note that the solution set is not a priori a vector space due to $g \neq 0$. One has therefore certainly not a linear parametrization between the solution set and $\mathbb{C}^2$.

Proof. the proof can be found in many ODE texts such as [11] (Thm 16.2). □

We proceed with some more notation toward the construction of some boundary triplet for $T^\star$.

Fix some compact interval $[\alpha, \beta] \subset (a,b)$ and consider the quantity

$$\int_\alpha^\beta [\mathscr{L}f(x)\overline{g(x)} - f(x)\overline{\mathscr{L}g(x)}]dx \tag{3.1}$$

with $f, g \in D(T^\star)$.
Using integration by parts and the fact that V is real (3.1) is equal to $[f,g]_\beta - [f,g]_\alpha$ with the shorthand notation $[f,g]_c := f(c)\overline{g^{'}(c)} - f^{'}(c)\overline{g(c)}$.
Because $f, g \in D(T^\star)$ the limits $[f,g]_a := lim_{\alpha \to a+0}[f,g]_\alpha$ and $[f,g]_b := lim_{\beta \to b-0}[f,g]_\beta$ are existing. Note that it does not mean that the functions themselves have some limits at the endpoints a, b.
One therefore writes

$$< T^\star f, g > - < f, T^\star g >= [f,g]_b - [f,g]_a \tag{3.2}$$

If $f \in D(T)$ or $g \in D(T)$ an immediate consequence of (3.2) is that $[f,g]_a = [f,g]_b = 0$.
To see this, consider that $f \in D(T)$ and choose $g_0 \in D(T^\star)$ wich is equal to g near one endpoint, say a, and 0 near the other.
With the help of standard smoothing procedure it is always possible to place oneself in such a situation.
The claim follows because $< T^\star f, g_0 > - < f, T^\star g_0 >= 0$ using the definition of the adjoint, $[f, g_0]_b = 0$ and $[f,g]_a = [f, g_0]_a$.

One is already in a position to say something about the defect numbers of the densely defined closed symmetric operator T.
Taking advantage of the fact that $dimN(T^\star - \lambda I) = dimN(T^\star - \overline{\lambda} I)$ because V is real-valued, Proposition 3.4 (2) forces this dimension to be either of the following three possibilities 0, 1 or 2.

In other words one has $(d_+, d_-) = (0,0), (1,1)$ or $(2,2)$.

One concludes that self-adjoint extensions for T are always existing.

More can be said if one assumes regularity at some endpoint.

One has the preliminary and important observation

Proposition 3.5. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then for each endpoint there exists a nonzero solution of*

$$\mathscr{L} f - \lambda f = 0 \tag{3.3}$$

which is L^2 near to that endpoint.

Proof. We carry out the proof for the endpoint b.
The idea is to restrict the operator T to some subinterval $(c,b) \subset (a,b)$.
One defines $T_c = T_{L^2(c,b)}$.
Considering $f, g \in C_0^\infty(a,b)$ with $f(c) = g'(c) = 0$ and $f'(c) = g(c) = 1$ and applying (3.2) to $T_c^\star$, one concludes that $T_c^\star$ is not symmetric hence that T_c is not self-adjoint (we saw earlier that T_c is symmetric).
This implies in particular that T_c has nonzero defect numbers.
It therefore exists a f_c satisfying $f_c \in L^2(c,b)$ and (3.3) on (c,b).
Now let $d \in (c,b)$ be arbitrary. By Proposition 3.4 (2) there exists a unique solution f of (3.3) (on (a,b)) satisfying $f(d) = f_c(d)$ and $f'(d) = f_c'(d)$. The unicity holding when restricted to the subinterval (c,d), one gets that $f = f_c$ on (c,b) and the proof is complete. □

Remark 3.6. In the preceding proof was used the fact that a non self-adjoint operator has non-zero defect numbers.
To see this one applies the von Neumann decomposition (2.7). The decomposition says exactly that a densely defined closed symmetric operator is self-adjoint iff its defect numbers are both 0.

The direct consequence is

Corollary 3.7. *If at least one end point is regular the defect numbers are* $(1,1)$ *or* $(2,2)$. *If both are regular we are in the* $(2,2)$ *case.*

We shall now introduce a slightly less restrictive notion than regularity at endpoints leading to some classification about the defect numbers of T.
The next theorem is due to H. Weyl

Theorem 3.8 (Weyl's alternative). *Let d be an endpoint of the interval (a,b). Then precisely one of the following two possibilities is valid:*

1. *For each $\lambda \in \mathbb{C}$ all solutions of $\mathscr{L} f - \lambda f = 0$ are L^2 near d.*
2. *for each $\lambda \in \mathbb{C}$ there exists one solution of $\mathscr{L} f - \lambda f = 0$ which is not in L^2 near d.*

Remark 3.9. 1. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and we are in alternative 3.8 (2), Proposition 3.5 implies that only one solution (modulo some constant) is L^2 near the endpoint d. Here one takes into account the vector space structure of the set of solutions of the above equation.

Proof. The proof is based on a clever use of the Volterra integral equation and can be found in [15] (Thm 15.8, p.348) for example. □

The latter theorem leads to the following definition

Definition 3.10. Case 1 of Theorem 3.8 is called the *limit circle case* at d. The other is called the *limit point case*.

For subsequent developments we give the following criteria for limit point case at infinity.

Proposition 3.11. *Suppose that $b = \infty$. If there are numbers $c \in (a,b)$ and $C > 0$ such that $V(x) \geq -Cx^2$ for all $x \in (c,\infty)$, then T is in the limit point case at $b = \infty$.*

Proof. See [2] (p.1407) for a proof. □

We continue to investigate what implies the Weyl's alternative concerning the defect numbers of T.

To obtain uselful conclusions a preliminary result need to be established.

Lemma 3.12. *Let d be a boundary point and suppose that T is in the limit point case at d. Then $[f,g]_d = 0$ for all $f, g \in D(T^\star)$.*

Proof. Let $d = b$.
The proof is again based on the consideration of the truncated operator T_c with $c \in (a,b)$ that is the restriction of T to $L^2(c,b)$.

We shall characterize the space $D(T_c^\star)$.

One has $T_c \subseteq T$ essentially because $C_0^\infty(c,d) \subset C_0^\infty(a,b)$.
This implies that $D(T^\star) \subseteq D(T_c^\star)$ (considering restrictions of the functions on (c,b)).

By the continuity of V on (a,b) T_c is in the circle case at c and in the limit point case at b. Therefore its defect numbers are $(1,1)$ and one gets, considering the von Neumann decomposition (2.7), that $dim D(T_c^\star)/D(T_c) = 2$.

Consider $f_1, f_2 \in C_0^\infty(a,b)$ with $f_1(c) = f_2^{'}(c) = 0$ and $f_1^{'}(c) = f_2(c) = 1$. By the paragraph just after Equation (3.2), one gets that $h(c) = h^{'}(c) = 0$ for $h \in D(T_c)$. Therefore one has $D(T_c^\star) = D(T_c) + Lin\{f_1, f_2\}$ with $Lin\{\}$ standing for linear span.
It thus follows that $[f,g]_b = [f_0, g_0]_b$ with f_0, g_0 being funtions such that $f - f_0$ and $g - g_0$ are vanishing at b. Finally $f_0, g_0 \in D(T_c)$ imply that $[f,g]_b = 0$ once more using the paragraph just after Equation (3.2).
The claim is now proved for functions in $D(T_c^\star)$ hence for functions in $D(T^\star)$. □

Theorem 3.13. *T has defect numbers as*

1. $(2,2)$ *if it is in the circle case at both endpoints*
2. $(1,1)$ *if it is in the circle case at one endpoint and in the limit point case at the other*
3. $(0,0)$ *if both endpoints are in the limit point case. Therefore T is already self-adjoint in that case.*

Proof. The only case which is not immediate is (3).
Using Lemma 3.12, one concludes that $T^\star$ is symmetric. This implies that T is self-adjoint because $T \subseteq T^\star \subseteq T^{\star\star} = T$, the first inclusion coming from the symmetry of T and the second one from symmetry of $T^\star$ and the last equality holding because T is closed. □

We finish by investigating the (existing) self-adjoint extensions of T using boundary triplet theory of Section 2.2. Those extensions strongly depend on the conditions we impose for the potential at the boundary points.
We shall only treat the case needed for our purpose.

The case of interest is that of regularity at endpoint a and limit point case at b.

By (3.2) and Lemma 3.12 we have that for $f, g \in D(T^\star)$

$$< T^\star f, g > - < f, T^\star g >= f^{'}(a)\overline{g(a)} - f(a)\overline{g^{'}(a)}$$

the functions f, g having continuous extensions to the boundary point a essentially due to the regularity of $\mathscr{L}$ at a using Proposition 3.4 (1).

One thus obtains a boundary triplet (whose existence is ensured by Theorem 2.25 because we are in the $(1,1)$ setting concerning the defect numbers) as

$$\mathcal{K} = \mathbb{C},\ \Gamma_0(f) = f(a),\ \Gamma_1(f) = f^{'}(a).$$

One applies Theorem 2.23 to find all self-adjoint extensions of T.
We parametrize the latter using self-adjoint operators $B \in S(\mathbb{C})$ as given in Theorem 2.23 (2) essentially because we are in the (Γ_0, Γ_1) form for the boundary triplet.

$\mathbb{C}$ having $\mathbb{C}$-dimension 1, $\mathbb{C}_B$ (the closed subspace of $\mathbb{C}$ on which B is defined) is either $\mathbb{C}$ or $\{0\}$.
In the $\mathbb{C}$ case, the self-adjoint B is just multiplication by a real number b.
In the $\{0\}$ case the operator B is trivially the self-adjoint $Id : \{0\} \to \{0\}$.

One therefore obtains the associated self-adjoint extensions of T corresponding to either cases as subspaces of $D(T^\star)$ described by

$$\begin{cases} f^{'}(a) = bf(a),\ b \in \mathbb{R} \\ f(a) = 0 \end{cases} \tag{3.4}$$

Remark 3.14.

1. Concerning the case $\mathbb{C}_B = \mathbb{C}$ and applying the representation of the self-adjoint relation related to the operator B given in Proposition 2.13 one observes that $\mathbb{C}_B^\perp = \{0\}$ and thus the self-adjoint extension for this case is completely characterized by the first line of (3.4).

2. In the case $\mathbb{C}_B = \{0\}$ one gets, again considering the representation of Proposition 2.13, that $f^{'}(a)$ is not constrained, that is $f^{'}(a) \in \mathbb{C}$. The self-adjoint extension for T is thus perfectly characterized by the second line in (3.4).

The shorthanding notation for (3.4) encodes all the possible self-adjoint extensions of T as

$$f(a)\cos\alpha = f^{'}(a)\sin\alpha,\ \alpha \in [0,\pi) \tag{3.5}$$

simply writing $b = \frac{1}{tan(\alpha)}$ with the given range for α. The case $\alpha = 0$ corresponds to $b = \infty$ and one views the second line in (3.4) to be represented by this case.

We end this section by showing the appearance of a function, the *Weyl-Titchmarsch* function, which will play a crucial role in the subsequent spectral theory for the self-adjoint extensions of T.

Because we are in the $(1,1)$ case concerning the defect numbers of T we know that for every

$\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is a unique (modulo constant) L^2 solution $\psi(.,\lambda)$ of $\mathscr{L}f - \lambda f = 0$.

If one is seeking for a decomposition of ψ in a fundamental system of solutions of $\mathscr{L}f - \lambda f = 0$, say $(\phi(.,\lambda), \theta(.,\lambda))$ such that $\phi(a,\lambda) = \theta^{'}(a,\lambda) = 0$ and $\phi^{'}(a,\lambda) = \theta(a,\lambda) = 1$, he has to take into account that ϕ and θ cannot then be L^2 functions because then they would be respectively in deficiency spaces of the self-adjoint extensions T_0 and T_1 given in Example 2.22 contradicting the self-adjointness of T_0, T_1.
This last observation leads, fixing arbitrarily but definitively the boundary values of ψ at a, to an expression of the form

$$\psi(.,\lambda) = \phi(.,\lambda) + m(\lambda)\theta(.,\lambda),$$

with $m(\lambda) \neq 0$.
The function m is called the *Weyl-Titchmarsch* function associated to the operator T.
We shall have the opportunity to develop some of its properties in the upcoming sections.

Herglotz functions

As we shall see in the next section, the function m mentionned at the end of section 3.1 will behave as a particular type of functions called Herglotz's or Nevanlinna's functions. Our goal in this section is to give some basic properties of the latter needed in the following. Most of the results presented is strongly connected with measure theory.

A Herglotz function f is an analytic function defined on the upper complex plane $\mathbb{C}_+$ with values in the upper complex plane that is $f : \mathbb{C}_+ \to \mathbb{C}_+$ with $\mathbb{C}_+ = \{z \in \mathbb{C} : Im(z) > 0\}$. Observe that each Herglotz function can be extended to an analytic function on the lower complex plane by setting $f(z) := \overline{f(\overline{z})}$. In general there is no analytic extension of f to $\mathbb{C}$.

We list the most important properties of such type of functions, almost all of them needed in the proof of the spectral theorem in the next section.

Theorem 3.15. *Let m be a Herglotz function. Then*

1. *$m(z)$ has finite normal limits $m(\lambda \pm i0) = lim_{\epsilon \to 0^+} m(\lambda \pm i\epsilon)$ for a.e. $\lambda \in \mathbb{R}$. Moreover $\epsilon |Re(m(\lambda \pm i\epsilon))| = o(1)$ as $\epsilon \to 0$ for every $\lambda \in \mathbb{R}$.*

2. *There exists a positive regular Borel measure $d\omega$ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1+\lambda^2} < \infty$ such that the following Riesz-Herglotz representation holds*

$$m(z) = c + dz + \int_{\mathbb{R}} d\omega(\lambda) [\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}]$$

with $c, d \in \mathbb{R}$ *and* $d \geq 0$.

3. *The measure* ω *can be recovered from* $Im(m(\lambda + i\epsilon))$ *in the following sense: if* $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, *then the Stieltjes inversion formula reads*

$$\omega((\lambda_1, \lambda_2]) = \pi^{-1} lim_{\delta \to 0^+} lim_{\epsilon \to 0^+} \int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda Im(m(\lambda + i\epsilon)). \tag{3.6}$$

For singleton one has for $\lambda \in \mathbb{R}$

$$\omega(\{\lambda\}) = lim_{\epsilon \to 0^+} \epsilon Im(m(\lambda + i\epsilon)) \tag{3.7}$$

4. m *has a representation as*

$$m(z) = \int_{\mathbb{R}} d\omega(\lambda)(\lambda - z)^{-1},$$

with ω *finite iff* $sup_{\eta>0} \eta |m(i\eta)| < \infty$. *In this case* $\omega(\mathbb{R})$ *is equal to the above* sup.

5. *The singularities of* m *on the real line are at most of the first order in the sense that: with the same notation than in (3) and with* $\eta_1 > 0$, *we have a constant* $C = C(\lambda_1, \lambda_2, \eta_1) > 0$ *such that*

$$\eta |m(\lambda + i\eta)| \leq C, \ (\lambda, \eta) \in [\lambda_1, \lambda_2] \times (0, \eta_1) \tag{3.8}$$

Proof. The proofs for (i) and (iii) are found in [1]. For (ii) one can consult [18] (Thm B.1, p.381). The latter proof is based on an application of the Cauchy integral representation theorem using cleverly the contour of integration.
For (iv) and (v) one finds the proofs in [5]. □

One has the following characterization concerning the decomposition of ω as $d\omega = d\omega_{sing} + d\omega_{ac}$ in singular and absolutely continuous parts

Theorem 3.16. 1. *The singular part of* ω, ω_{sing}, *is supported by*

$$S_{\omega_{sing}} = \{t \in \mathbb{R} : Im(m)(t + i0) = \infty\}$$

2. *The absolute continuous part* ω_{ac} *is given by*

$$d\omega_{ac}(t) = \pi^{-1}(Im(m))(t + i0)dt$$

and supported by

$$S_{\omega_{ac}} = \{t \in \mathbb{R} : 0 < Im(m)(t + i0) < \infty\}$$

Proof. It uses the de la Vallee Poussin theorem which enables to relate the symmetric derivative of ω with $\pi^{-1}(Im(m))(t + i0)dt$.

When one shows the representation

$$d\omega_{ac}(t) = \pi^{-1}(Im(m))(t+i0)dt$$

for the absolute part of ω, its support $S_{\omega_{ac}}$ is obvious considering that $(Im(m))(t+i0)$ has to be in $L^1(\mathbb{R})$ relative to the Lebesgue measure.
Complete proof can be found in [15] (Thm F.6, p.414) □

The following result specializes in the case of a purely absolutely continuous measure.

Theorem 3.17. *Suppose that*

$$sup\{Im(m)(z) : z \in \mathbb{C}_+\} < \infty \tag{3.9}$$

and that the sup condition in Proposition 3.15 (4) is satisfied.

Then the function $Im(m)(t+i0)$ is in $L^1(\mathbb{R})$ and we have

$$m(z) = \pi^{-1}\int_{\mathbb{R}} d\lambda \frac{Im(m(\lambda+i0))}{\lambda - z}, \tag{3.10}$$

that is the measure associated to m is purely absolutely continuous.

Proof. By (3.9) and (3.7) we get that $\omega(\{c\}) = 0$ for all $c \in \mathbb{R}$. Now by (3.6) one writes

$$\omega((a,b)) = \pi^{-1} lim_{\epsilon\to 0^+}\int_a^b d\lambda Im(m(\lambda+i\epsilon)) = \int_a^b d\lambda Im(m(\lambda+i0)) \tag{3.11}$$

by Lebesgue integral theorem using again (3.9).
The measure ω being finite by hypothesis one obtains the L^1 property stated by (3.11).
The representation for m given in (3.10) follows since the purely absolutely continuous measure ω satisfies then the expression

$$d\omega(t) = \pi^{-1} Im(m(t+i0))dt.$$

□

Spectral Theorem

The goal of this section is to prove the spectral theorem for a Sturm-Liouville operator T whose operator expression $\mathscr{L}$ is regular at a and which is in the limit point case at ∞.

The interval of interest for this section is the half-line $[a,\infty)$.

We consider the following self-adjoint extension T_0 of T (remember that T is the closure of the operator expression $\mathscr{L} = -\frac{d^2}{dx^2} + V(x)$ on C_0^∞)

$$\begin{aligned} &T_0 f = \mathscr{L} f \\ &D(T_0) = \{f \in L^2(a,b) : f, f' \in AC[\alpha,\beta] \text{ for } [\alpha,\beta] \subset (a,b), \\ &\mathscr{L} f \in L^2(a,b),\ f(a) = 0\} \end{aligned} \tag{3.12}$$

Note that this is the extension found in (3.5) with $\alpha = 0$. This is also the extension given in Example 2.22.
The functions in the domain are clearly continuous up to the boundary a by Proposition 3.4 (1). Therefore $D(T_0)$ makes sense.

Next one introduces the standard fundamental system of solutions $\phi(.,z)$ and $\theta(.,z)$, $z \in \mathbb{C}$ of

$$\mathscr{L}\psi(x,z) = z\psi(x,z) \tag{3.13}$$

satisfying the initial conditions $\phi(a,z) = \theta'(a,z) = 0$ and $\phi'(a,z) = \theta(a,z) = 1$, whose unicity and existence is ensured by Proposition 3.4. Those two solutions are then spanning the whole set of solutions of (3.13). As seen above they are not L^2 functions. They are analytic in z by some use of the Volterra integral equation.
Considering the endpoints hypothesis and Proposition 3.5 one gets for $z \in \mathbb{C} \setminus \mathbb{R}$ the existence of a unique L^2 solution $\psi_+(.,z)$ of (3.13) satisfying $\psi_+(a,z) = 1$ written as

$$\psi_+(.,z) = \theta(.,z) + m(z)\phi(.,z), \tag{3.14}$$

with $m \neq 0$ being the Weyl-Titchmarsch's function associated to T.
m will be shortly shown to be a Herglotz function.

One then introduces the Green function for T_0, which gives one the possibilty to obtain some tractable expression for the resolvents associated to T_0.

The basic idea behind the following representation for the Green function is to find a way to inverse the operator $T_0 - z$ with $z \in \mathbb{C} \setminus \mathbb{R}$, which is known to be invertible due to the self-adjointness of T_0.
If one considers its action on $D(T_0)$ as $f \mapsto (T_0 - z)f = (\mathscr{L} - z)f$ one wants to find a way to get back to f.
A solution is coming using the Dirac 'function' which lets one to write

$$f(x) = \int_a^\infty f(x')\delta(x - x')dx'. \tag{3.15}$$

Therefore if one is able to find a function $G_0(x,x',z)$, $(x,x') \in (a,\infty) \times (a,\infty)$, such that for any fixed $x \in (a,\infty)$ one has

$$(\mathscr{L}_{x'} - z)G_0 = \delta(x - x') \tag{3.16}$$

he will obtain a further representation for f as

$$f(x) = \int_a^\infty f(x')(\mathscr{L}_{x'} - z)G_0 dx' \tag{3.17}$$

which can be transformed using integration by parts to the seeked representation for f as

$$f(x) = \int_a^\infty (\mathscr{L}_{x'} - z)f(x')G_0 dx'. \tag{3.18}$$

Moreover G_0 has to be continuous on its domain, the only non-trivial points to be tested for continuity being when $x = x'$. One sees that this is indeed the case essentially considering integration (in x') of the equality (3.16) near x and using the properties of the Dirac.

Of course a careful treatment needs to deal with a number of technical difficulties. For example the Dirac 'function' is not strictly speaking a function but a distribution. Further integration by parts without boundary terms at ∞ is only a priori possible with functions vanishing at infinity. Therefore one will have certainly to work with $C_0^\infty(a,\infty)$ functions and use density of the latter in $L^2(a,\infty)$ to obtain a solid argumentation.

In the present case G_0 is given by

$$G_0(x,x',z) = \begin{cases} \phi(x,z)\psi_+(x',z), \ a \leq x \leq x' \\ \phi(x',z)\psi_+(x,z), \ a \leq x' \leq x \end{cases} \tag{3.19}$$

and one gets the formula for the resolvents of T_0 as

$$((T_0 - zI)^{-1}f)(x) = \int_a^\infty dx' G_0(x,x',z)f(x'), \ f \in L^2([a,\infty),dx) \tag{3.20}$$

We remark that the integral is well-defined because $\psi_+ \in L^2(a,\infty)$ and we also obtain a function in $D(T_0)$ because $G(a,x',z) = 0$ and ϕ, ψ_+ and their first order derivatives being AC on compacts subintervals of (a,∞).

For concluding to the Herglotz's nature of m one begins with the following lemma

Lemma 3.18. *With $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$, $z_1 \neq z_2$ we have the following*

$$\int_a^\infty dx \psi_+(x,z_1)\psi_+(x,z_2) = \frac{m(z_1) - m(z_2)}{z_1 - z_2} \tag{3.21}$$

Proof. This will be a consequence of the identity

$$\frac{d}{dx}W(\psi(.,z_1),\psi(.,z_2))(x) = (z_1 - z_2)\psi(x,z_1)\psi(x,z_2) \tag{3.22}$$

where $\psi(.,z_1)$ and $\psi(.,z_2)$ are two arbitrary solutions of (3.13) associated to z_1 and z_2 respectively and $W(f,g)$ is the Wronskian of the two functions f and g.
Note that $lim_{x\to\infty}W(\psi_+(.,z_1),\psi_+(.,z_2))(x) = 0$ using Lemma 3.12 combined with the fact that $\psi_+(.,z_1),\psi_+(.,z_2) \in D(T^\star)$.

It remains to prove the identity (3.22).
For simplicity we write $\psi_1 := \psi(.,z_1)$ and $\psi_2 := \psi(.,z_2)$ and observe that

$$W(\psi_1,\psi_2) = W(\psi_1,\psi_2 - \psi_1).$$

Now $\psi_2 - \psi_1$ satisfies

$$\mathscr{L}(\psi_2 - \psi_1) - z_1(\psi_2 - \psi_1) = (z_2 - z_1)\psi_2.$$

We can therefore use the Volterra integral equation to write

$$\begin{aligned}&(\psi_2 - \psi_1)(x) = \\ &\phi_1(x)\int_{x_0}^{x} dx'\theta_1(x')(z_1 - z_2)\psi_2(x') - \theta_1(x)\int_{x_0}^{x} dx'\phi_1(x')(z_1 - z_2)\psi_2(x'),\end{aligned} \tag{3.23}$$

where x_0 is an arbitrary point in $[a,\infty)$ and $\phi_1 := \phi(.,z_1), \theta_1 := \theta(.,z_1)$ is the already introduced fundamental system of solutions of (3.13) for z_1.
By simple computations one finally gets the result by using that $W(\phi_1,\theta_1)(x) = 1$ and the fact that $W(\psi_1,\phi_1) = W(\psi_1,\theta_1) =$ constant. □

One then obtains

Proposition 3.19. *m is an Herglotz function*

Proof. Since $\overline{\psi_+(.,z)}$ is, due to the fact that V is real-valued, a L^2 solution of $\mathscr{L}\psi(x,\overline{z}) = \overline{z}\psi(x,\overline{z})$ satisfying $\psi(a,z) = 1$ one conlcudes that $\overline{\psi_+(.,z)} = \psi_+(.,\overline{z})$.
The same being true for the fundamental system of solutions at any $z \in \mathbb{C}$, one ends up with

$$\overline{m(z)} = m(\overline{z}). \tag{3.24}$$

Plugging this in (3.21) one obtains

$$\int_a^\infty dx|\psi_+(x,z)|^2 = \frac{Im(m(z))}{Im(z)} \tag{3.25}$$

It therefore remain to prove analyticity of m on $\mathbb{C}_+$ for the proof being complete.

The trick is to write the action of the resolvent at $z \in \mathbb{C}_+$ in two different ways, one of them using the Green function at z.
Let c and d be such that $a \leq c < d < \infty$. Then using (3.19), (3.20) and basic spectral theory one writes for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}&\int_{\sigma(T_0)} \frac{d\|E_{T_0}(\lambda)\chi_{[c,d]}\|^2_{L^2([a,\infty))}}{\lambda - z} =< \chi_{[c,d]}, (T_0 - zI)^{-1}\chi_{[c,d]} > \\ &= \int_c^d dx \int_c^x dx^{'} \theta(x,z)\phi(x^{'},z) + \int_c^d dx \int_x^d dx^{'} \phi(x,z)\theta(x^{'},z) + \\ &m(z)[\int_c^d dx \phi(x,z)]^2\end{aligned} \tag{3.26}$$

with $\chi_{[c,d]}$ being the indicator function of the given interval and $E_{T_0}(\lambda)$ the spectral projection measure associated to the self-adjoint T_0.
Since the left-hand side of (3.26), $\theta(x,z)$ and $\phi(x,z)$ are all analytic in z, observing the finiteness of the measure on the left-hand side of (3.20) and the compact integration range for the two integrals on the right-hand side of (3.20), m will be analytic if one is able to divide by $[\int_c^d dx\phi(x,z)]^2$ for some well-chosen range for $z \in \mathbb{C}_+$.
Consider $z_0 \in \mathbb{C}_+$ arbitrary and observe that, using that $\phi(x,z_0)$ is nonzero combined with the continuity in x and the analyticity in z of $\phi(x,z)$, one can choose $c(z_0), d(z_0) \in [a,\infty)$ and a neighborhood $V_{z_0} \subset \mathbb{C}_+$ of z_0 such that $[\int_{c(z_0)}^{d(z_0)} dx\phi(x,z)]^2 \neq 0$ for all $z \in V_{x_0}$.
One then concludes to the analyticity in this V_{z_0} and therefore on all $\mathbb{C}_+$, z_0 being arbitrary. □

Remark 3.20.

1. We observe that the range for z could have been extended to $\mathbb{C} \setminus \sigma(T_0)$ without destroying the analyticity, noting that $\sigma(T_0)$ is closed.
   One thus concludes that m can be analytically extended to this larger domain.
   We also point out that this extension for m satisfies the expected property that $m(\eta) \in \mathbb{R}$ if $\eta \in \mathbb{R} \setminus \sigma(T_0)$ beacuse ϕ and θ have this same property by the first paragraph in the proof of (3.19).

2. Specific representation for m as seen in Theorem 3.15 could be determined but we do not need it in the following.
   The only property which is interesting for subsequent developments is that, as is seen in Equation (3.7), $lim_{\epsilon \to 0^+} \epsilon Im(m(\lambda + i\epsilon))$ is detecting the pure point support of the associated measure and this is reflected in the present case for m as one gets the following from (3.25)

$$lim_{\epsilon \to 0^+} \epsilon Im(m(\lambda + i\epsilon)) = \begin{cases} 0, & \phi(.,\lambda) \notin L^2([a,\infty)) \\ \|\phi(.,\lambda)\|_{L^2([a,\infty))}, & \phi(.,\lambda) \in L^2([a,\infty)). \end{cases} \tag{3.27}$$

We shall now relate the spectral measure $E_{T_0}(\lambda)$ to the measure associated to m.

The use of the Stone formula in weak form (see [2] (p.1203) for a proof)

$$< f, F(T_0)E_{T_0}((\lambda_1,\lambda_2])g >=$$
$$lim_{\delta\to 0} lim_{\epsilon\to 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda F(\lambda) < \left(< f, R_{\lambda+i\epsilon}(T_0)g > - < f, R_{\lambda-i\epsilon}(T_0)g >\right), \tag{3.28}$$

with $f,g \in L^2((0,\infty),dr)$, $F \in C(\mathbb{R})$ and $\lambda_1 < \lambda_2 \in \mathbb{R}$,
enables one to express the sepctral (projection) measure $\{E_{T_0}(\lambda) : \lambda \in \mathbb{R}\}$ associated to the self-adjoint T_0 by means of the resolvents of T_0. The latter are bounded operators defined on the whole $L^2(a,\infty)$ and are therefore more tractable specifically in the present case using the Green function representation (3.20).
Complementing the latter considerations about the Stone formula with the properties of the function m as described in Theorem 3.15 for dealing with limit processes and inversions of integrals, one is able to prove the next technical proposition which will naturally lead to the proof of the spectral theorem.

Proposition 3.21. *Let $f,g \in C_0^\infty((a,\infty))$, $F \in C(\mathbb{R})$, and $\lambda_1,\lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$. Then*

$$< f, F(T_0)E_{T_0}((\lambda_1,\lambda_2])g >_{L^2([a,\infty),dx)} =< \hat{f}, M_F M_{\chi_{(\lambda_1,\lambda_2]}}\hat{g} >_{L^2(\mathbb{R},d\rho)}, \tag{3.29}$$

where we have defined

$$\hat{h}(\lambda) = \int_a^\infty dx\phi(x,\lambda)h(x),\ \lambda \in \mathbb{R},\ h \in C_0^\infty((a,\infty)), \tag{3.30}$$

and M_G, for G measurable, is the standard multiplication operator on $L^2(\mathbb{R},d\rho)$ with ρ being the measure associated to m.

Proof. See [6] (Thm 2.6, p.7) for a proof. □

Remark 3.22.
1. One observes that the weak operator equality (3.29) is valid specifically for C_0^∞ functions. This is in fact crucial to consider such type of functions for being able to prove the former equality. We shall see shortly how one can extend it to L^2 functions

2. The left-hand side of (3.29) is well defined because, using basic operator theory, $E_{T_0}((\lambda_1,\lambda_2])g \in D(F(T_0))$.

We come now to the so-called Distorted Fourier representation which basically says that the $\hat{(.)}$-transform (3.30) is a unitary operator from $L^2([a,\infty),dx)$ to $L^2(\mathbb{R},d\rho)$.

One first remarks that, by considering $F = 1$ and letting $\lambda_1 \to -\infty$ and $\lambda_2 \to \infty$ in (3.29), one concludes that the $\hat{(.)}$-transform is an isometry from $C_0^\infty((a,\infty))$, equipped with L^2-norm, into $L^2(\mathbb{R},d\rho)$. One can thus, by density argument, extend it to an isometry U_0 from $L^2([a,\infty))$ into $L^2(\mathbb{R},d\rho)$.

The question of the surjectivity of U_0 is answered by the following (whose proof, adapted in the present context, is given in greater generality in [6] (pp. 10-13))

Theorem 3.23 (Distorted Fourier representation)**.** *The map*

$$U_0 : \begin{cases} L^2([a,\infty)) \to L^2(\mathbb{R}, d\rho) \\ h \mapsto \hat{h}(.) = lim_{b\to\infty} \int_a^b dx\phi(x,.)h(x) \end{cases} \tag{3.31}$$

where the right-hand side limit stands for $L^2(\mathbb{R}, d\rho)$-limit is onto (and therefore unitary).

Proof. Considering (3.29) with $F = 1$, an application of Fubini's theorem gives (the triple integral on the right-hand side of (3.32) can be written in any order due to the compact support of the functions, the analycity of $\phi(x,.)$ and the finite measure domain of integration in λ)

$$\begin{aligned} &< f, E_{T_0}((\lambda_1,\lambda_2])g >_{L^2([a,\infty),dx)} \\ &= \int_a^\infty dx\overline{f(x)} \int_a^\infty dx^{'} g(x^{'}) \int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\phi(x,\lambda)\phi(x^{'},\lambda) \end{aligned} \tag{3.32}$$

which leads to

$$E_{T_0}((\lambda_1,\lambda_2])g = \int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\phi(x,\lambda)\hat{g}(\lambda) \tag{3.33}$$

valid for $g \in C_0^\infty([a,\infty))$.
Equality (3.33) extends by continuity to all $L^2([a,\infty))$. On the left-hand side of (3.33) one uses the boundedness of the projector E_{T_0} and for the right-hand side the isometry property of the $\hat{(.)}$ transform coupled to the subsequence converging a.e. propoerty of the L^2 convergence and a use of Fubini's theorem due to finite measure domain of integration in λ permits to conclude.

By taking successive $L^2([a,\infty), dx)$-limits as $\lambda_1 \to -\infty$ and $\lambda_2 \to \infty$ on both sides of (3.33) (writing the left-hand side of (3.33) as $E_{T_0}(\lambda_2)g - E_{T_0}(\lambda_1)g$ with $E_{T_0}(\lambda) = E_{T_0}((-\infty,\lambda])$ and taking into account the well-known strong limits for the spectral projection measure) one gets

$$g = lim_{\lambda_1\to-\infty,\, \lambda_2\to\infty} \int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\phi(x,\lambda)\hat{g}(\lambda),\ g \in L^2([a,\infty)) \tag{3.34}$$

The latter equality gives a candidate for the inverse of U_0 as one obtains $\tilde{V}_0 U_0 = Id_{L^2([a,\infty))}$ with

$$\tilde{V}_0 : \begin{cases} D_{max} \to L^2([a,\infty)), \\ \hat{h} \mapsto lim_{\lambda_1\to-\infty,\, \lambda_1\to\infty} \int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\phi(.,\lambda)\hat{h}(\lambda), \end{cases} \tag{3.35}$$

the right-hand side limit considered as a $L^2([a,\infty))$-limit and $D_{max} \subseteq L^2(\mathbb{R}, d\rho)$ indicating the maximal domain for the map in question being well-defind.
It contains in particular $U_0(L^2([a,\infty)))$.

The proof will be complete if one can show that $\tilde{V_0}$ is bounded, injective and that $C_0^\infty(\mathbb{R}) \subset D_{max}$.
If this can be achieved, using the uniform continuity of $\tilde{V_0}$, the density of $C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R}, d\rho)$ and the fact that $L^2([a,\infty))$ is a Banach space, the set D_{max} is $L^2(\mathbb{R}, d\rho)$.
Moreover $\tilde{V_0}$ is onto considering $\tilde{V_0} U_0 = Id_{L^2([a,\infty))}$.

U_0 will then be unitary with bounded inverse given by $\tilde{V_0}$.

Pick some $\hat{f} \in C_0^\infty(\mathbb{R})$.
Let $g \in C_0^\infty((a,\infty))$. The goal is to show that $\tilde{V_0}(\hat{f})$ has a sense or is well-defined. Looking at the definition of $\tilde{V_0}$ one notes that, by an abuse of notation, $\tilde{V_0}(\hat{f})$ is a well-defined measurable function (but one is not certain that the latter function is a member of $L^2((a,\infty))$).
To achieve the proof that $\tilde{V_0}(\hat{f})$ is indeed defined one has to prove that its $L^2[a,\infty)$-norm is finite.
By the use of Fubini's Theorem one obtains

$$< g, \tilde{V_0}(\hat{f}) >_{L^2[a,\infty),dx} = < U_0 g, \hat{f} >_{L^2(\mathbb{R},d\rho)} . \tag{3.36}$$

Thus

$$\|\tilde{V_0}(\hat{f})\|_{L^2([a,\infty))} = sup_{g\in C_0^\infty([a,\infty)),\, g\neq 0} \left| \frac{< g, \tilde{V_0}(\hat{f}) >}{\|g\|_{L^2([a,\infty))}} \right|$$

and the right-hand side of the latter is $\leq \|\hat{f}\|_{L^2(\mathbb{R},d\rho)}$ using that U_0 is isometry and (3.36).
This concludes the proof that $C_0^\infty(\mathbb{R}) \subset D_{max}$ and that $\tilde{V_0}$ is bounded considering density. One therefore concludes that $D_{max} = L^2(\mathbb{R}, d\rho)$.

One comes finally to the proof of the injectivity of $\tilde{V_0}$. He shall take advantage of the action of the resolvents of T_0 eventually obtaining a Borel transform of a regular complex measure enabling him to apply Theorem 3.15.
The heart of the proof lies in the following identity

$$\tilde{V_0}((.-z)^{-1}\hat{f}) = (T_0 - z)^{-1}\tilde{V_0}\hat{f},\ z \in \mathbb{C}_+,\ \hat{f} \in L^2(\mathbb{R}, d\rho). \tag{3.37}$$

We shall prove that, with $\hat{f} \in L^2(\mathbb{R}, d\rho)$, $\lambda_1 < \lambda_2 \in \mathbb{R}$, we have

$$(T_0 - z)\int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\frac{\phi(.,\lambda)\hat{f}(\lambda)}{\lambda - z} = \int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\phi(.,\lambda)\hat{f}(\lambda). \tag{3.38}$$

Equality (3.37) readily follows by taking $L^2([a,\infty), dx)$-limits as $\lambda_1 \to -\infty$ and $\lambda_2 \to \infty$.

One first observes that $\int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\frac{\phi(a,\lambda)\hat{f}(\lambda)}{\lambda-z} = 0$ by the definition of $\phi(.,\lambda)$.
Therefore $\int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\frac{\phi(.,\lambda)\hat{f}(\lambda)}{\lambda-z} \in D(T_0)$.

We then express the left-hand side of (3.38) as

$$(\mathscr{L}-z)\int_{\lambda_1}^{\lambda_2} d\rho(\lambda)\frac{\phi(.,\lambda)\hat{f}(\lambda)}{\lambda-z}$$

and consider plugging this operator expression inside the integral, valid in the present context due to the finite measure range of integration in λ. The property of ϕ gives one the right-hand side of (3.38).

Now suppose $\hat{f}_0 \in Ker(\tilde{V}_0)$ and consider a sequence $(\hat{f}_n)_n \subset C_0^\infty(\mathbb{R})$ such that $\|\hat{f}_n - \hat{f}_0\|_{L^2(\mathbb{R},d\rho)} \to 0$ as $n \to \infty$.
Applying Equality (3.37) to each $\hat{f}_n$ and integrating on both side on the interval $[a, y]$ with $y \in [a,\infty)$ one gets

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda-z}(\int_a^y dx\phi(x,\lambda))\hat{f}_n(\lambda) = \int_a^y dx((T_0-z)^{-1}\tilde{V}_0\hat{f}_n)(x), \tag{3.39}$$

where an application of the Fubini's Theorem has been performed on the left-hand side again valid due to the compact range of integration in (x,λ).

One then taked limits as $n \to \infty$ on both sides of (3.39).
For the left-hand side one uses that $\int_a^y dx\phi(x,\lambda) \in L^2(\mathbb{R}, d\rho)$ because it can be written as $U_0(\chi_{[a,y]})$.
For the right-hand side one takes advantage of the boundedness of the two operators $(T_0-z)^{-1}$ and $\tilde{V}_0$ combined with the continuous inclusion $L^2([a, y], dx) \subseteq L^1([a, y], dx)$ to obtain

$$\int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda-z}(\int_a^y dx\phi(x,\lambda))\hat{f}_0(\lambda) = \int_a^y dx((T_0-z)^{-1}\tilde{V}_0\hat{f}_0)(x). \tag{3.40}$$

The right-hand side of (3.40) is then 0 and we are thus left on the left-hand side of (3.40), with a Stieltjes (or Borel) transform for the regular complex Borel measure $d\rho(\lambda)(\int_a^y dx\phi(x,\lambda))\hat{f}_0(\lambda)$ (remark the product of the two $L^2(\mathbb{R}, d\rho)$-functions $\int_a^y dx\phi(x,\lambda)$ and $\hat{f}_0(\lambda)$).
Applying Theorem 3.15 (3) implies that

$$\int_{(\lambda_1,\lambda_2]} d\rho(\lambda)(\int_a^y dx\phi(x,\lambda))\hat{f}_0(\lambda) = 0. \tag{3.41}$$

One finally considers second derivative in y of the preceding equality obtaining (remark the valid use of the Lebesgue derivative Theorem due to the finite measure range of integration in λ)

$$\int_{(\lambda_1,\lambda_2]} d\rho(\lambda)\phi_y(y,\lambda)\hat{f}_0(\lambda) = 0. \tag{3.42}$$

Taking in the last equation $y = a$ one gets

$$\int_{(\lambda_1,\lambda_2]} d\rho(\lambda)\hat{f}_0(\lambda) = 0. \tag{3.43}$$

Because the interval of integration is arbitrary one therefore concludes that $\hat{f}_0 = 0$ ρ-a.e. on $\mathbb{R}$ finishing the proof. □

We end this subsection with the spectral theorem for T_0 which is a direct consequence of Theorem 3.23

Theorem 3.24 (Spectral Theorem)**.** *With the same notations as in Proposition 3.21 one gets*

$$U_0 F(T_0) U_0^{-1} = M_F \text{ on } L^2(\mathbb{R}, d\rho) \tag{3.44}$$

Moreover the spectra of the operators written as functions of T_0 are given by

$$\begin{aligned} \sigma(T_0) &= supp(d\rho) \\ \sigma(F(T_0)) &= ess.ran_{d\rho}(F) \end{aligned} \tag{3.45}$$

Proof. Considering (3.29)

$$< f, F(T_0)E_{T_0}((\lambda_1,\lambda_2])g >_{L^2([a,\infty),dx)} = < \hat{f}, M_F M_{\chi_{(\lambda_1,\lambda_2]}} \hat{g} >_{L^2(\mathbb{R},d\rho)}$$

and taking advantage that $\bigcup_{\{M\,:\,M \text{ Borel set}\}} E_{T_0}(M) C_0^\infty(a,\infty)$ is a core for $F(T_0)$ one reduces the latter equality to

$$< f, F(T_0)g >_{L^2([a,\infty),dx)} = < \hat{f}, M_F \hat{g} >_{L^2(\mathbb{R},d\rho)}$$

with $g \in D(F(T_0))$ and $f \in L^2([a,\infty))$.
One concludes the proof using the unitarity of U_0 shown in Theorem 3.23.

The expressions for the spectra are given using the invariance by unitarity along with basic spectral theory of multiplication operators. □

Remark 3.25. 1.

$$ess.ran_{d\rho}(F) = \{z \in \mathbb{C} : \forall \epsilon > 0,\ \rho(\{\lambda \in \mathbb{R} : |F(\lambda) - z| < \epsilon\}) > 0\} \tag{3.46}$$

2. (3.44) gives T_0 as multiplication by λ (modulo a unitary) and therefore T_0 has simple spectrum. This particularly imply that all its eigenvalues have multiplicity one.

4 Fourier Basis

We shall describe in details the Fourier basis for the operator in concern. Most of the results in this chapter were obtained in (see [10]) and thus will only be stated without giving full proofs. The specific representation for the Jost solution of (3.13) associated to the operator in concern for the rest of the text will nevertheless be proved in great details as it represents one of the main tool for performig subsequent calculations.

Linearized Operator

If one considers the critical wave equation in dimension 3 as

$$\Box\psi - \psi^5 = 0$$

(in radial setting) and linearizes it around the stationary radial Aubin-Talenti solution $W := \phi(.,1) = (1+\frac{r^2}{3})^{-\frac{1}{2}}$ as in (1.4), that is if one considers radial solutions of the form $\psi = W + u(t,r)$, the radiative part u will then satisfy the following

$$\partial_{tt}u + Hu = N(u,W). \tag{4.1}$$

H is given by $H = -\triangle - 5W^4 = -\triangle + V$ and is called the linearized Hamiltonian associated to the static solution W. It has some potential part V given by

$$V = -\frac{5}{(1+\frac{r^2}{3})^2} \tag{4.2}$$

Finally N represents the nonlinearity term whose form is similar (with obvious modifications) to the one given in (1.7).

We remove the nonlinearity for the moment, being primarily interested in the analysis of the linearized operator H.

If one considers v of the form $v = r\,u$, where u is solution of $\partial_{tt}u + Hu = 0$, then v will satisfy the following equation

$$\partial_{tt}v + \mathscr{L}v = 0 \tag{4.3}$$

with $\mathscr{L}$ given by

$$\mathscr{L} = -\partial_{rr} - \frac{5}{(1+\frac{r^2}{3})^2}. \tag{4.4}$$

Performing the transformation $U : u \to r\,u$ was meant to remove the ∂_r-term in the Laplacian recovering a Sturm-Liouville form for the linearized operator $\mathscr{L}$.
One notes that the spectral properties of H and $\mathscr{L}$ are the same if he considers $\mathscr{L}$ acting on $L^2((0,\infty))$ because U is unitary from $L^2_{rad}(\mathbb{R}^3)$ to $L^2((0,\infty))$.
One therefore says that acting with U is passing from space dimension 3 to space dimension 1.

$\mathscr{L}$ being in the limit circle case at 0, due to regularity, and in the limit point case at ∞ by proposition 3.11, has self-adjoint extensions given by (3.5).
We consider specifically the self-adjoint extension T_0 (remember that one has to consider the (existing!) symmetric closure of $\mathscr{L}$ denoted by T) given by (3.12) with $\alpha = 0$.

One can argue that this specific self-adjoint extension is considered because one wants to get $D(S) = U(D(H))$ for the self-adjoint extension S. In other words the domain for S has to be the image under U of the domain of the self-adjoint H with domain $D(H)$. Moreover one knows that, $\mathscr{L}$ being regular at 0, the functions from $D(S)$ will have continuous extension to 0.
If one sums up all the preceding considerations, the value at 0 of a member v of $D(S)$ has to satisfy $v(0) = r\,u(0) = 0$.

In the preceding paragraph and for the rest of the text H is considered self-adjoint with domain $D(H)$. Basic spectral theory of differential operators ensures the existence of such a $D(H)$ essentially because the potential part V is a compact perturbation of the Laplacian (in dimension 3).

The main concern in this section will be to give precise expressions for the Fourier basis associated to $\mathscr{L}$. This is the solution $\phi(r,z)$ of (3.13) with prescribed boundary values as $\phi(0,z) = 0$ and $\phi'(0,z) = 1$. By self-adjointness of T_0 it is obvious that $\phi(r,z) \notin L^2([0,\infty))$ if $z \in \mathbb{C}\setminus\mathbb{R}$.

Before giving the detailed results concerning the specific form of the Fourier basis, one explains how he can estimate the spectral measure associated to T_0.
This shall be done using an additional solution of (3.13), the so-called Jost solution, denoted $f_+(r,z)$, $Im(z) \geq 0$.

This solution is particularly tractable as one knows (seen below) its behaviour when $r \to \infty$.
A supplementary advantage when dealing with the Jost solution is given by the fact that $f_+(r,\xi)$ and its complex conjugate $f_-(r,\xi) := \overline{f_+(r,\xi)}$ form a basis for the solution set of (3.13) when $z = \xi \in \mathbb{R}_+$.
Decomposing $\phi(.,\xi)$ in the latter basis will result in what is called the *Jost representation* for the Fourier basis ϕ (more details below).

One can show, using the Volterra integral representation combined with some recursive construction, that a solution characterized by the following two conditions exists (and moreover is analytic in $\mathbb{C}_+$):

$$\begin{cases} |f_+(r,z) - \exp i\sqrt{z}r| \to 0 \text{ as } r \to \infty,\ Im(z) \geq 0 \\ Im(\sqrt{z}) \geq 0 \end{cases} \tag{4.5}$$

Note that the second condition implies that $f_+(r,z) \in L^2([0,\infty))$ if $Im(z) > 0$. Moreover one has $f_+(r,\xi + i\epsilon) \to \exp i\xi^{1/2} r$ as $\epsilon \to 0$, if $\xi > 0$ and $r \to \infty$. One then shows that there exists $c(\xi) \neq 0$ a.e. in $\xi > 0$ such that

$$\psi_+(r,\xi + i0) \sim c(\xi) \exp i\xi^{1/2} r,\ r \to \infty. \tag{4.6}$$

To see the latter one uses the unicity of the L^2 solution when $Im(z) > 0$ and the fact that $\psi_+(r,\xi + i\epsilon)$ has a well-defined limit as $\epsilon \to 0$ a.e. in $\xi > 0$ considering the analycity of the fundamental system of solutions of (3.13) given by the usual ϕ (Fourier basis) and θ and also the Herglotz property of m given by 3.15 (1).
Here ψ_+ is the existing L^2 solution of (3.13) given under the form (3.14).

The fact that the representation (4.6) holds only for $\xi > 0$ is no issue for subsequent developments because the essential spectrum of T_0 (or H) is $[0,\infty)$.
Moreover one shall only consider action of T_0 on the essential spectrum projected part of the domain (see Chapter 5).

We shall see shortly that the essential spectrum of $\mathscr{L}$ is purely absolutely continuous.

It is consequently only on this interval that one needs the precise form for the spectral measure, which will therefore be given by a spectral density w.r.t. the Lebesgue measure on $[0,\infty)$.

Using the Jost solution enables one to find the normal limits of the Weyl-Titchmarsch function

m appearing in (3.14) considering the equality

$$m(\xi + i0) = \frac{W(\theta(.,\xi), f_+(.,\xi))}{W(f_+(.,\xi), \phi(.,\xi))}, \tag{4.7}$$

where the link between the Jost solution f_+ and ψ_+ is given by $\psi_+(r,z) = c(z) f_+$, $Im(z) > 0$ together with the normal limit behaviour given in (4.6).

It happens that one does not need any expression for θ as it cancels in subsequent calculations. We shall therefore only concentrate on finfding tractable expressions for the Fourier basis ϕ.

One gets

Proposition 4.1.

$$\phi(r,z) = \phi_0(r) + r^{-1} \sum_{j=1}^{\infty} (r^2 z)^j \phi_j(r^2) \tag{4.8}$$

where

$$\phi_0(r) = r(1 - r^2/3)(1 + r^2/3)^{-3/2} \tag{4.9}$$

is the resonance term satisfying $\phi_0(r) = \phi(r,0)$ *and* $\phi_j(u)$ *is real-analytic for* $u > 0$ *and is bounded as*

$$|\phi_j(u)| \leq \frac{C^j}{(j-1)!} u < u >^{-1} \tag{4.10}$$

For $\xi > 0$ *the Jost solutions can be written in the following form*

$$f_+(r,\xi) = \exp i r \xi^{1/2} \sigma(r\xi^{1/2}, r) \tag{4.11}$$

where σ *has the symbolic asymptotic sum representation*

$$\sigma(q,r) \sim \sum_{j=0}^{\infty} q^{-j} \psi_j^+(r) \tag{4.12}$$

in the following sense

$$\begin{cases} sup_{r>0} < r >^2 |(r\partial_r)^\alpha (q\partial_q)^\beta [\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j} \psi_j^+(r)]| \leq c_{\alpha,\beta,j_0} q^{-j_0-1} \\ \textit{for all } \alpha, \beta, j_0 \geq 0 \end{cases} \tag{4.13}$$

where $\psi_0^+(r) = 1$ *and the* $\psi_j^+(r)$ *are symbol of order* -2 *satisfying*

$$sup_{r>0} < r >^2 |(r\partial_r)^\alpha \psi_j^+(r)| < \infty$$

for all $\alpha \geq 0$ *and* $j \geq 1$.

Remark 4.2. 1. The series on the right-had side of (4.8) is absolutely converging for all r, z being bounded by an exponential series using (4.10). One can therefore use this

expression on the whole domain $[0,\infty) \times (0,\infty)$.
As will be seen in Chapter 5 when obtaining linear dispersive estimates for the propagation operators related to T_0, the expression for ϕ given in (4.8) will not be sufficient. Hence is the need for some other representation for the Fourier basis which will be obtained using the Jost solution (see Proposition 4.13 below).

2. We shall come back to the notion of symbols before (see Subsection 4.2.1) proving the expression (4.11).

3. One observes that (4.13) is only true when $q > 1$ and thus will be used only in the regime $r^2\xi > 1$.

Proof. The proof for (4.8) through (4.10) can be found in [10] (Prop 4.4, p.36).

As was already mentionned in the introduction for this chapter, we postpone the proof of the representation for the Jost solution (4.11) to the next section. □

Jost representation

Symbolic Asymptotic Sum Representation

We describe very briefly the theory of symbols needed for constructing an *asymptotic symbolic sum.*

Let $X \subset \mathbb{R}^n$, $n \geq 0$, be an open subset and let $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$, $m \in \mathbb{R}$ and $N \in \mathbb{N} \setminus 0$.
Symbols are defined by

Definition 4.3. The space of symbols of order m and of type (ρ,δ), written $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, is the space of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all compact set $K \subset X$ and all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, there exists some constant $C = C_{K,\alpha,\beta}(a)$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}, \qquad (x,\theta) \in K \times \mathbb{R}^N.$$

Such a space is a naturally a Frechet space toplogized by the 'sup' seminorms given by

$$P_{K,\alpha,\beta}(a) = sup_{(x,\theta)\in K\times\mathbb{R}^N} \frac{|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)|}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}},$$

K running on all compact subsets of X and α,β as in the above Definition.
This space is metrizable as one can choose a countable set of such seminorms giving the same topology on $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ essentially considering a compact exhaustion for X.

A simple example is the following

Example 4.4. $e^{ix\xi}$, with scalar product meaning for $x\xi$, is in $S^0_{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$.

One introduces an additional set of functions, noted $S^{-\infty}$, as

Definition 4.5. $S^{-\infty}(X\times\mathbb{R}^N)$ is the set of all $a \in C^\infty(X\times\mathbb{R}^N)$ such that, with the same notations as in Definition 4.3, and with $M \in \mathbb{R}$, one has the existence of a constant $C = C_{K,\alpha,\beta,M}(a)$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C(1+|\theta|)^M, \qquad (x,\theta) \in K\times\mathbb{R}^N.$$

$S^{-\infty}$ is thus the space of symbols with all possible decays for every derivatives (with one of the two variables contained in some compact).
It is also a Frechet (metrizable) space and one writes, for fixed (ρ,δ) as in Definition 4.3,

$$S^{-\infty} = \cap_{m\in\mathbb{R}} S^m_{\rho,\delta} \tag{4.14}$$

Remark 4.6. There is no point of introducing symbols of type other than with $(\rho,\delta) \in [0,1] \times [0,1]$.
To see this in a special case, let's consider $m < 0$ and $\rho > 1$.
It is then easy to obtain that a symbol with such parameters would be contained in $S^{-\infty}$. One just applies derivative operators under the form $\sum_i \theta_i \partial_{\theta i}$ followed by intergration (performing essentially integration by parts whose boundary terms will be removed due to the value of m and the form of the derivative operator).

The following three propositions are found and proved in [7] (pp.7-8).
We give here the technical proofs as it is almost impossible to clearly understand the construction of the symbolic asymptotic sum (see below) without dealing with them.

Proposition 4.7. *Let $(a_j)_{j=1}^\infty$ be a bounded sequence in $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ which converges pointwise to a. Then a belongs to $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ and for every $m' > m$, we have $a_j \to a$ in $S^{m'}_{\rho,\delta}(X\times\mathbb{R}^N)$ (that is in the topology of $S^{m'}_{\rho,\delta}(X\times\mathbb{R}^N)$).*

Remark 4.8. We clearly have $S^m_{\rho,\delta}(X\times\mathbb{R}^N) \subset S^{m'}_{\rho,\delta}(X\times\mathbb{R}^N)$ if $m' > m$, the embedding being continuous. In other words, on $S^m_{\rho,\delta}$ the topology of $S^{m'}_{\rho,\delta}$ is weaker than that of $S^m_{\rho,\delta}$.

Proof. If $f \in C^2([-\epsilon,\epsilon])$ with $\epsilon > 0$, then one has

$$|f'(0)| \le C_\epsilon(\|f\|^{1/2}_{L^\infty}\|f''\|^{1/2}_{L^\infty} + \|f\|^{1/2}_{L^\infty}) \tag{4.15}$$

as an application of Taylor expansion. Applying (4.15) recursively for every derivatives in each variable (restricting the domain of the sequence $(a_i)_i$ to some compact of $X\times\mathbb{R}^N$) and

using the fact that the sequence $(a_j)_j$ is bounded in $S^m_{\rho,\delta}$, one concludes that $(a_j)_j$ is Cauchy in $C^\infty(X\times\mathbb{R}^N)$ and thus converges to some $a\in C^\infty(X\times\mathbb{R}^N)$, which lies in $S^m_{\rho,\delta}$, again taking advantage of the boundedness of $(a_j)_j$.
One is left to show convergence to this a in the topology of $S^{m'}_{\rho,\delta}$.

Fixing some compact $K\subset X$, one has to show that the following quantity converges uniformly on $(x,\theta)\in K\times\mathbb{R}^N$ to 0

$$k_j(x,\theta)=\frac{|\partial_x^\alpha\partial_\theta^\beta(a_j-a)(x,\theta)|}{(1+|\theta|)^{m'-\rho|\beta|+\delta|\alpha|}}. \tag{4.16}$$

Using that $m'>m$ one writes (4.16) as

$$\frac{1}{(1+|\theta|)^{m'-m}}\frac{|\partial_x^\alpha\partial_\theta^\beta(a_j-a)(x,\theta)|}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}} \tag{4.17}$$

Fix $\epsilon>0$ (not the same ϵ as in the beginning of the proof).
First one can choose some R_ϵ such that if $|\theta|\geq R_\epsilon$ (4.17) is uniformly (in $j,x\in K,\theta$) bounded by ϵ remembering that $(a_j)_j$ is bounded in $S^m_{\rho,\delta}$.
On the other hand if $|\theta|\leq R_\epsilon$, just argue by uniform convergence of (a_j) to a on compacts of $X\times\mathbb{R}^N$. □

In the construction of the asymptotic sum we shall need the following

Proposition 4.9. *$S^{-\infty}(X\times\mathbb{R}^N)$ is dense in $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ in the toplogy of $S^{m'}_{\rho,\delta}(X\times\mathbb{R}^N)$*

Proof. Given an element $a\in S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ one constructs a sequence $(a_j)_j\subset S^{-\infty}(X\times\mathbb{R}^N)$ converging to a in the topology of $S^{m'}_{\rho,\delta}(X\times\mathbb{R}^N)$.
By Proposition 4.7 it suffices to check for the pointwise convergence and boundedness in $S^m_{\rho,\delta}$ of the sequence $(a_j)_j$.
We shall use a standard cutoff construction.

Let $\chi(\theta)$ be $C_0^\infty(\mathbb{R}^N)$ such that $\chi=1$ on $\{|\theta|\leq 1\}$ and $\chi=0$ when $|\theta|>2$. One then defines $\chi_j(\theta)=\chi(\frac{\theta}{j})$ and considers the sequence $(a_j)_j\subset S^{-\infty}$ given by $a_j=\chi_j(\theta)a(x,\theta)$.
We show that $(a_j)_j$ satisfies the above requirements.
The pointwise convergence to a is clear.
To show that $(a_j)_j$ is bounded in $S^m_{\rho,\delta}$, consider the equality $|\partial^\alpha\chi_j(\theta)|=j^{-|\alpha|}|\partial^\alpha\chi(\frac{\theta}{j})|$, noting that the latter has support in $\{j\leq|\theta|\leq 2j\}$. This leads to $(\chi_j)_j\subset S^0_{1,0}$ is bounded. Observing that $S^0_{1,0}\subset S^0_{\rho,\delta}$ and taking advantage of the fact that the bilinear map

$$\begin{cases} S^0_{\rho,\delta}\times S^m_{\rho,\delta}\to S^m_{\rho,\delta}\\ (a,b)\mapsto ab\end{cases} \tag{4.18}$$

is continuous (essentially an application of Leibniz formula) permits to conclude the proof. □

One finally obtains

Proposition 4.10. *[Asymptotic Sum Existence] Let* $a_j \in S^{m_j}_{\rho,\delta}$ *with* $m_j \to -\infty$ *as* $j \to \infty$*. Then there exists* $a \in S^{m_0}_{\rho,\delta}$ *(unique modulo* $S^{-\infty}$*) such that*

$$a - \sum_{0\le j<k} a_j \in S^{m_k}_{\rho,\delta}, \quad k \ge 0 \tag{4.19}$$

Proof. The unicity modulo $S^{-\infty}$ is due to (4.14) and the behaviour of the sequence $(m_j)_j$.
For proving the existence of a one uses the Cantor diagonalization process as follows.
For each $j \ge 0$ let $(P_{j,l})_{l\ge 0}$ be a sequence of seminorms giving the topology on $S^{m_j}_{\rho,\delta}$. Now by Proposition 4.9 one is able to find for each $j > 0$ a $b_j \in S^{-\infty}$ such that $P_{\mu,\nu}(a_j - b_j) < 2^{-j}$ for $0 \le \mu, \nu \le j-1$. By using the Cantor's process the series $\sum_{j\ge k}(a_j - b_j)$ is converging in $S^{m_k}_{\rho,\delta}$ for all $k \ge 0$. In particular one gets that $a := \sum_{j\ge 0}(a_j - b_j) \in S^{m_0}_{\rho,\delta}$.
To check for (4.19) one writes

$$a - \sum_{0\le j<k} a_j = -\sum_{0\le j<k} b_j + \sum_{j\ge k}(a_j - b_j) \in S^{m_k}_{\rho,\delta}$$

□

Remark 4.11. Due to Proposition 4.10 we shall work from now on with the symbolic spaces in quotient form $S^m_{\rho,\delta}/S^{-\infty}$

Definition 4.12. The a constructed in Proposition 4.10 is called the *asymptotic sum* of the $(a_j)_j$ and noted shortly as $a \sim \sum_{j=1}^{\infty} a_j$

Proof of Proposition 4.1

We are now ready to prove the main result of this chapter concerning the representation for the Jost solution f_+.

Proof of proposition 4.1 (4.11)-(4.13): plugging the expression (4.11) into the equation

$$\mathscr{L} f_+ = z f_+$$

one writes the differential equation σ has to satisfy in the form

$$(-\partial_{rr} - 2i\xi^{1/2}\partial_r - \frac{5}{(1+r^2/3)^2})\sigma(r\xi^{1/2}, r) = 0 \tag{4.20}$$

remarking that the differential operators in the last expression are acting exclusively on the second argument of $\sigma(r\xi^{1/2}, r)$, considering therefore the $r\xi^{1/2}$ as a variable in its own right.

We call this variable the *regime*.
Consider for σ the following formal power series form as

$$\sum_{j=0}^{\infty} \xi^{-j/2} f_j(r) \tag{4.21}$$

yielding a recurrence relation for $(f_j)_j$ given by

$$\begin{aligned} 2i\partial_r f_j &= (-\partial_{rr} - \frac{5}{(1+r^2/3)^2}) f_{j-1} \\ f_j(\infty) &= f_j^{'}(\infty) = 0 \end{aligned} \tag{4.22}$$

where we imposed Cauchy values at ∞ for being able to integrate the latter relation.
Beginning with $f_0 = 1$ one solves for f_j, $j \geq 1$, as

$$f_j(r) = \frac{i}{2}\partial_r f_{j-1}(r) - \frac{i}{2}\int_r^{\infty} dr^{'} (\frac{5}{(1+r^{'2}/3)^2}) f_{j-1}(r^{'}). \tag{4.23}$$

As $r \to \infty$ it is easy to obtain an asymptotic expressions (using Taylor development) for $f_j(r)$ as

$$f_j(r) = O(r^{-j-2}). \tag{4.24}$$

Symbolic behaviour for derivatives is obtained in exactly the same way, by deriving (4.23) and using again Taylor expansions when $r \to \infty$.

Defining

$$\psi_j^+ := r^j f_j(r), \tag{4.25}$$

we get the desired behaviour for the ψ_j^+ as symbols of order -2 for $j \geq 1$.
Note also that when $r \to 0$ one obtains a smooth behaviour for the ψ_j^+ and a bound in $O(1)$ using recursively (4.23).

We know from Proposition 4.10 that there exists σ_{as} such that for any $K \subset (0,\infty)$ compact, there exists a positive $c_{K,\alpha,\beta,j_0} \geq 0$ such that

$$\begin{aligned} &sup_{r\in K}|(r\partial_r)^{\alpha}(q\partial_q)^{\beta}[\sigma_{as}(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)]| \\ &\leq c_{K,\alpha,\beta,j_0} q^{-j_0-1} \end{aligned} \tag{4.26}$$

holds for $q > 1$, $j_0 \geq 0$, essentially because one has

$$q^{-j}\psi_j^+ \in S_{1,0}^{-j}((0,\infty)\times(1,\infty)),\ j \geq 0$$

using separation of variables behaviour and the fact that $\psi_j^+(r)$ are symbols of order -2 (in the

r-variable only).

To be more precise one considers the precise form of σ_{as} as given by Proposition 4.10

$$\sum_{j=0}^{\infty} q^{-j}\psi_j^+(r)\chi(q\delta_j), \tag{4.27}$$

χ being some cutoff function in $C^\infty((0,\infty))$ with $supp(\chi)\subset(a,\infty)$, $\chi_{[b,\infty)}=1$ with $0<a<b<\infty$ and the sequence $(\delta_j)_j$ converging to 0 such that

$$\sum_{j=k}^{\infty} q^{-j}\psi_j^+(r)\chi(q\delta_j) \tag{4.28}$$

converges in $S_{1,0}^{-k}((0,\infty)\times(1,\infty))$, $k\geq 0$.
Note the fact that it is crucial for this latter convergence to take place that $q\geq 1$ as is shown by (4.17).
Moreover if one restricts the variable q to the open $(1,\infty)$, the convergence in the r variable for the sum (4.28) is uniform (due to the specific form for the ψ_j^+) and therefore one can get rid of the compact K in expression (4.26) and even absorb the symbolic order (in r variable) of the ψ_j^+ writing, with some constant $c_{\alpha,\beta,j_0}>0$ not anymore depending on K,

$$\begin{aligned} &sup_{r\in(0,\infty)} <r>^2 |(r\partial_r)^\alpha (q\partial_q)^\beta[\sigma_{as}(q,r)-\sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)]| \\ &\leq c_{\alpha,\beta,j_0} q^{-j_0-1}. \end{aligned} \tag{4.29}$$

(4.29) is the most tractable expression for controlling the behaviour of σ_{as} in the (r,q)-variables.

The last step is to show that $\sigma-\sigma_{as}\in S^{-\infty}((0,\infty)\times(1,\infty))$.

We introduce the error

$$e(q,r)=(-\partial_{rr}-2i\xi^{1/2}\partial_r-\frac{5}{(1+r^2/3)^2})\sigma_{as}(q,r) \tag{4.30}$$

where it is clear using (4.29) and observing that the formal infinite sum (4.21) cut at some finite index is solution of (4.20) that e satisfies the following

$$<r>^4 |(r\partial_r)^\alpha (q\partial_q)^\beta e(q,r)| \leq c_{\alpha,\beta,j} q^{-j} \tag{4.31}$$

for every $j\geq 0$, that is e has $S^{-\infty}$ behaviour in q with some controlled decay in r of order 4.

One therefore defines the difference $\sigma_1 = \sigma_{as} - \sigma$ satisfying

$$(-\partial_{rr} - 2i\xi^{1/2}\partial_r - \frac{5}{(1+r^2/3)^2})\sigma_1(q,r) = e(q,r) \tag{4.32}$$

and imposing some Cauchy conditions at infinity, we shall prove that σ_1 has $S^{-\infty}$ behaviour in q with controlled decay in r-variable.
As above we impose 0 Cauchy conditions at infinity (in r variable) as $\sigma_1(q,\infty) = \sigma_1^{'}(q,\infty) = 0$ and will prove that

$$< r >^2 |(r\partial_r)^\alpha (q\partial_q)^\beta \sigma_1(q,r)| \leq c_{\alpha,\beta,j} q^{-j} \tag{4.33}$$

for each $j \geq 2$. The proof will then be finished.

The idea is to transform the second order equation (4.32) into a first order system and then use the Gronwall lemma in a clever way. We write (4.32) as

$$\partial_r \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & r^{-1} \\ -\frac{5}{(1+r^2/3)^2} & r^{-1} - 2i\xi^{1/2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ re \end{pmatrix} \tag{4.34}$$

with $v := (v_1, v_2) = (\sigma_1, r\partial_r\sigma_1)$ which can be written in compact matrix form as

$$\partial_r v - Av = b,$$

where $A := \begin{pmatrix} 0 & r^{-1} \\ -\frac{5}{(1+r^2/3)^2} & r^{-1} - 2i\xi^{1/2} \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ re \end{pmatrix}$.

Working now at the $\mathbb{C}^2$ level, we get $\partial_r|v|^2 = \partial_r < v, v >= \partial_r(v\overline{v}) = (Av)\overline{v} + b\overline{v} + (\overline{A}\overline{v})v + b\overline{v}$ with obvious meaning for the scalar product and overline notations. Performing some simple algebraic manipulations we finally get $|\partial_r|v|^2| \leq r^{-1}|v|^2 + r|v||e|$. It was important to pass to vector notation and then taking derivatives of the norm of v for being able to get rid of the $2i\xi^{1/2}$ term in A. We therefore have $\partial_r|v|^2 \geq -r^{-1}|v|^2 - r|v||e|$ and because $\partial_r|v|^2 = 2|v|\partial_r|v|$, we are finally left with

$$\partial_r|v| \geq -r^{-1}|v| - r|e|. \tag{4.35}$$

Integrating (4.35), using the Cauchy conditions, one obtains

$$|v|(r) \leq \int_r^\infty (r^{'-1}|v|)dr^{'} + \alpha(r) \tag{4.36}$$

with $\alpha(q,r) = \int_r^\infty r^{'}|e|(q,r^{'})dr^{'}$
(we note that all our functions are depending on (q,r) but from now on only the r dependance will be indicated).

Defining

$$y(s) = \exp(\int_1^s dr^{'}\frac{1}{r^{'}})(-\int_s^\infty dr^{'}\frac{1}{r^{'}}|v|(r^{'})) \tag{4.37}$$

and remarking that

$$|y(s)| = s\int_s^\infty dr' \frac{1}{r'}|v|(r') \le \int_s^\infty dr' |v|(r') \to 0, \quad s \to \infty. \tag{4.38}$$

one derives obtaining the inequality

$$\begin{aligned} y'(s) = \exp(\int_1^s dr' \frac{1}{r'})\frac{1}{s}(-\int_s^\infty (\frac{1}{r'}|v|(r'))dr' + |v|(s)) \\ \le \exp(\int_1^s dr' \frac{1}{r'})\frac{1}{s}\alpha(s) \end{aligned} \tag{4.39}$$

because $\exp(\int_1^s dr' \frac{1}{r'})\frac{1}{s} > 0$ and by using (4.36). Now, by integrating $\int_r^\infty ds y'(s) = -y(r)$ (using (4.38)), one gets $-y(r) \le \int_r^\infty ds \exp(\int_1^s dr' \frac{1}{r'})\frac{1}{s}\alpha(s)$ and finally using the definition of $y(r)$ and $\exp(\int_1^r dr' \frac{1}{r'}) = r$ one concludes that

$$\int_r^\infty dr' \frac{1}{r'}|v|(r') \le r^{-1}\int_r^\infty ds\alpha(s). \tag{4.40}$$

Therefore, by (4.36) and (4.40), we get

$$|v(r)| \le \alpha(r) + r^{-1}\int_r^\infty ds\alpha(s) \tag{4.41}$$

ending the adaptation of the proof of the Gronwall lemma in the present context.

By using (4.31) and the definition of α, one therefore gets that

$$< r >^2 |v(q,r)| \le C_j q^{-j}, \quad j \ge 2. \tag{4.42}$$

It thus remains to argue for the higher order derivatives.
We shall see what happens with first order derivatives of v, the general case being obtained by induction using the same pattern.

We first investigate what happens with the term $r\partial_r v$. Commuting with the matrix operator A (in other words writing the differential equation satisfied by $r\partial_r v$ as a first order system with the same form as in (4.34)) one gets

$$\partial_r(r\partial_r)v - A(r\partial_r)v = Av + b + r(\partial_r A)v + r\partial_r b. \tag{4.43}$$

The symbolic behaviour on the right-hand side of (4.43) is known by what has been established in (4.42), the only additional information needed here being that $|\partial_r A|$ is as $O(r^{-1})$. By running exactly the same procedure as for v one obtains the desired behaviour for $r\partial_r v$. For more details on the whole iterative procedure one can consult [10] (p.40).

□

Spectral Measure

We are now in a position to estimate the form of the spectral measure of the operator T_0.

Before giving the results about the spectral measure, one introduces

$$f_-(r,\xi) := \overline{f_+(r,\xi)}$$

and observes that for $\xi > 0$, $f_+(r,\xi)$ and $f_-(r,\xi)$ are two independant solutions of (3.13) because their Wronskian is equal to $-2i\xi^{1/2}$. We can then decompose $\phi(.,\xi)$ with respect to this basis as

$$\phi(r,\xi) = a(\xi) f_+(r,\xi) + \overline{a(\xi)} f_-(r,\xi) \tag{4.44}$$

the $\overline{a(\xi)}$ appearing because $\phi(r,\xi) \in \mathbb{R}$.

We observed in (4.7) the relation between the Fourier basis, the Jost solution and the spectral measure (in fact its Herglotz associated function).

(4.7) is true for any z with $Im(z) > 0$, that is

$$m(z) = \frac{W(\theta(.,z), f_+(.,z))}{W(f_+(.,z), \phi(.,z))}$$

so that if one can show the finiteness, for every $\xi \in (0,\infty)$, of the expression

$$Im\left(\frac{W(\theta(.,\xi), f_+(.,\xi))}{W(f_+(.,\xi), \phi(.,\xi))}\right) \tag{4.45}$$

this will then imply, if the latter limit is strictly positive, that the spectral measure of T_0 is strictly absolutely continuous on $(0,\infty)$ by an application of Theorem 3.16.
We would therefore denote its spectral density $\rho(\xi)$.

One can also conclude to the latter behaviour for the spectral measure on (0∞) by observing if the condition (3.9) in Theorem 3.17 is satisfied.
In fact if one additionally shows the continuity of (4.45) in $(0,\infty)$, he will therefore obtain the uniform convergence on every compact interval $[\xi_1,\xi_2]$, $0 < \xi_1 < \xi_2$ of the expression

$$Im\left(\frac{W(\theta(.,\xi+i\epsilon), f_+(.,\xi+i\epsilon))}{W(f_+(.,\xi+i\epsilon), \phi(.,\xi+i\epsilon))}\right)$$

and thus in a (complex) neighborhood of every such interval the condition (3.17) will be satisfied. One thus concludes to the pure absolute continuity of the spectral measure on $(0,\infty)$ because the compact intervals generate the Borel sigma algebra on the positive real axis.

It thus suffices to find the precise form for the absolute continuous density of the measure associated to m, given by $\rho = Im(m(\xi + i0))$, in the range $\xi \to \infty$ and $\xi \to 0$.

Applying the techniques used in [10], one is able to write $\rho(\xi)$ as

$$\rho(\xi) = \frac{1}{\pi} Im(m(\xi + i0)) = \frac{\xi^{1/2}}{\pi |W(\phi(.,\xi), f_+(.,\xi))|^2} \tag{4.46}$$

essentially by taking benefit of the fact that

$$W(f_+(.,\xi), f_-(.,\xi)) = -2i\xi^{1/2},$$

thanks to the form of Jost solution as $r \to \infty$ and the expansion of f_- and f_+ in ϕ, θ basis.

At the same time one finds the decomposition of $\phi(.,\xi)$ in f_+, f_- basis as in (4.44). The coefficient $a(\xi)$ appearing in this decomposition is then given by

$$a(\xi) = \frac{W(\phi(.,\xi), f_-(.,\xi))}{-2i\xi^{1/2}}. \tag{4.47}$$

One thus concludes that it remains to estimate the quantity

$$W(\phi(.,\xi), f_+(.,\xi)).$$

Omitting the technical calculations one finally obtains the following asymptotic behaviour concerning the spectral density and the coefficient a

Proposition 4.13. *The density of the absolute continuous part of the measure associated to m (the spectral measure of our T_0) satisfies*

$$\rho(\xi) \asymp \begin{cases} \xi^{-1/2} \; as\, \xi \to 0 \\ \xi^{1/2} \; as\, \xi \to \infty \end{cases} \tag{4.48}$$

meaning they are equivalent in norm. Moreover the derivatives have symbolic behaviour meaning that we can just write for ρ' the same equivalence as in (4.48) *but with standard derivatives for the right-hand side.*

The coefficients $a(\xi)$ satisfies

$$|a(\xi)| \asymp \begin{cases} 1 \; as \, \xi \to 0 \\ \xi^{-1/2} \; as \, \xi \to \infty \end{cases} \tag{4.49}$$

with similar symbolic behaviour.

Proof. See [10] (Lemma 4.6, p.41) for a proof □

5 Linear Dispersive Esimates

In the following chapter, when showing the existence of the stable manifold for the solutions of (1.1) linearized around some stationary Aubin-Talenti solution, we shall need to have certain control on some evolution operators related to the corresponding linearized operator.

We are treating in this chapter a model case resulting from linearization around Aubin-Talenti solution with $\lambda = 1$.

Consider we are working on $L^2_{rad}(\mathbb{R}^3)$ for the entire chapter.

The Hamiltonian H resulting from linearization around the Aubin-Talenti solution for $\lambda = 1$ was already seen to be the differential operator given by $H = -\triangle - 5W^4$ where $W = (1+\frac{r^2}{3})^{-1/2}$.

The evolution operators associated to H under consideration are given by

$$A(t) := cos(t\sqrt{HP_c}), \quad t > 0 \tag{5.1}$$

and

$$B(t) := \frac{sin(t\sqrt{HP_c})}{\sqrt{HP_c}}, \quad t > 0, \tag{5.2}$$

with P_c being the projector on continuous spectrum of H.
One has to consider the (self-adjoint) projection on the continuous spectrum of the operator H given by $P_c H P_c = HP_c$ because this projection is positive and thus the square root $\sqrt{HP_c}$ is well-defined.

Note that $B(t)$ is well-defined in dimension $n = 3$ because the continuous spectrum of H is equal to its essential spectrum using the fact that 0 is resonance.

Working exclusively on the continuous spectrum of H in the following developments we can lighten notation and write $A(t)$ and $B(t)$ as

$$A(t) = cos(t\sqrt{H}),$$
$$B(t) = \frac{sin(t\sqrt{H})}{\sqrt{H}}, \tag{5.3}$$

observing that $A(t)$ is clearly bounded from $P_c L^2_{rad}(\mathbb{R}^n)$ to itself.

The goal of this chapter is to obtain linear dispersive estimates for $A(t)$ and $B(t)$, that is some control of the $\|.\|_\infty$-norm of the image (action under operator) by a time decay t^{-a} with $a > 0$ multiplied by some Sobolev norm of the starting function.

Cosine Evolution

We shall begin by making some observations about the operator $A(t)$.

Using the theory developed in subsection 3.3.1, one writes the action of the operator $A(t)$ on $u_0 \in P_c\mathscr{D}(H)$ as

$$A(t)u_0 = \frac{1}{r}\int_0^\infty cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)\rho(\xi)d\xi, \tag{5.4}$$

where ϕ is the Fourier basis whose behaviour is given in Propositions 4.1 and 4.13 and ρ is the absolute continuous spectral density of H given by (4.48).
Moreover $v_0(\xi) = \widehat{ru_0}(\xi)$, the Fourier transform of ru_0, recalling that one has to pass in dimension 1 by multiplying by r before taking Fourier transform.
The inverse operation (recovering dimension 3), which is multiplication by $1/r$, will reveal to be crucial for finding linear dispersive estimates, as one knows that there is no dispersion in dimension 1.

Our goal is then to obtain some time decay bounding of $\|A(t)u_0\|_\infty := \|A(t)u_0\|_{L^\infty(\mathbb{R}^3)}$ by finding an adequate Sobolev space for u_0.

We shall prove the following

Lemma 5.1. *Denoting $C > 0$ some constant not depending on u_0, $A(t)$ satisfies the following linear dispersive estimate*

$$\|A(t)u_0\|_\infty \le C\frac{1}{t}\| < r > u_0\|_{H^{5/2(+)}(\mathbb{R}^3,dx)} \tag{5.5}$$

Remark 5.2.

1. One observes the free decay form (in dimension 3) for this linear dispersive estimate. In other words one is recovering the free decay in this linearized setting.

2. the weight $< r >$ on the right-hand side of (5.1) is needed because the method for obtaining the estimate, based on an integration by parts (IP) technique, will ask for the Sobolev control of derivatives of $v_0(\xi)$.

Proof. To separate the ranges $\xi \to 0$ and $\xi \to \infty$ we introduce some cutoff $\chi \in C_0^\infty(\mathbb{R}_+)$ with $\chi = 1$ on $[0,\epsilon_1]$, $0 \le \chi \le 1$ and $supp(\chi) \subset [0,\epsilon_2)$ where $1 \gg \epsilon_2 > \epsilon_1 > 0$.

We can therefore approximate the right-hand side of (5.4), using (4.48), by

$$\begin{aligned} &\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2}) v_0(\xi)\phi(r,\xi)\chi(\xi)\xi^{-1/2} d\xi \, + \\ &\frac{1}{r}\int_{\epsilon_2}^{\infty} cos(t\xi^{1/2}) v_0(\xi)\phi(r,\xi)(1-\chi)\xi^{1/2} d\xi \, , \end{aligned} \tag{5.6}$$

The idea is now to bound the $\|.\|_\infty$ norm of those two terms performing some integration by part, taking advantage of the argument of cos.

Considering the first term on the right-hand side of (5.6), it is easy to observe that one can write it as

$$\begin{aligned} &\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2}) v_0(\xi)\phi(r,\xi)\chi(\xi)\xi^{-1/2} d\xi = \\ &\frac{1}{t}\frac{1}{r}\int_0^{\epsilon_2} \partial_\xi(sin(t\xi^{1/2})) v_0(\xi)\phi(r,\xi)\chi(\xi) d\xi \, , \end{aligned} \tag{5.7}$$

showing up the factor t^{-1}.
By performing an integration by parts (IP) in (5.7), one gets

$$\begin{aligned} &\frac{1}{t}\frac{1}{r}\int_0^{\epsilon_2} \partial_\xi(sin(t\xi^{1/2})) v_0(\xi)\phi(r,\xi)\chi(\xi) d\xi = \\ &\frac{1}{t}\frac{1}{r}\int_0^{\epsilon_2} sin(t\xi^{1/2}) \partial_\xi \left(v_0(\xi)\phi(r,\xi)\chi(\xi)\right) d\xi \end{aligned} \tag{5.8}$$

observing that the boundary terms are trivially 0 using essentially the properties of the cutoff χ.

If one considers the term coming from $\phi_\xi := \partial_\xi \phi$ and, using (4.8), performs a derivative in ξ he will obtain

$$\begin{aligned}
&\phi_\xi = r^{-1}\left(r^2\xi\phi_1(r^2) + r\sum_{j\geq 2}(r^2\xi)^j\phi_j(r^2)\right)_\xi = \\
&r\phi_1(r^2) + r^3\sum_{j\geq 2} j(r^2\xi)^{j-1}\phi_j(r^2) = \\
&r\phi_1(r^2) + r^3(r^2\xi)\sum_{j\geq 2}(j-1)(r^2\xi)^{j-2}\phi_j(r^2) + r^3\sum_{j\geq 2}(r^2\xi)^{j-1}\phi_j(r^2),
\end{aligned} \tag{5.9}$$

both sums on the last line of (5.9) being absolutely convergent on whole domain $(r,\xi) \in (0,\infty)\times(0,\infty)$ using (4.10).

At this stage one immediately observes, even without taking care of the precise values of the sums in (5.9), that there will be too much positive power of r implying the impossibility to bound $\|.\|_\infty$.
One also notes that he is not allowed to use the expression for ϕ using the Jost solution of (3.13) because he has not specified the regime variable q (see (4.12)) as $r\xi^{1/2} > 1$.

To avoid this difficulty the idea is to introduce a second cutoff controlling the $r^2\xi$ term for then being able to use the Jost representation of ϕ.
One therefore introduces $\tilde{\chi} \in C_0^\infty(\mathbb{R}_+)$ with $\tilde{\chi} = 1$ on $[0, 1+\tilde{\epsilon_1}]$, $0 \leq \chi \leq 1$ and $supp(\tilde{\chi}) \subset [0, 1+\tilde{\epsilon_2})$ with $1 \gg \tilde{\epsilon_2} > \tilde{\epsilon_1} > 0$ then splitting once more each member in (5.6) obtaining the following four parts expression

$$\begin{aligned}
&\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)\chi(\xi)\tilde{\chi}(r^2\xi)\xi^{-1/2}d\xi + \\
&\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)\chi(\xi)(1-\tilde{\chi}(r^2\xi))\xi^{-1/2}d\xi + \\
&\frac{1}{r}\int_{\epsilon_2}^{\infty} cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)(1-\chi)(\xi)\tilde{\chi}(r^2\xi)\xi^{1/2}d\xi + \\
&\frac{1}{r}\int_{\epsilon_2}^{\infty} cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)(1-\chi)(\xi)(1-\tilde{\chi}(r^2\xi))\xi^{1/2}d\xi
\end{aligned} \tag{5.10}$$

which will be abreviated respectively as the $\chi\tilde{\chi}$, $\chi(1-\tilde{\chi})$, $(1-\chi)\tilde{\chi}$ and $(1-\chi)(1-\tilde{\chi})$-terms.
In the following developments, essentially for lightening the notation, we shall only write $\chi := \chi(\xi)$ and $\tilde{\chi} := \tilde{\chi}(r^2\xi)$ writing arguments only when confusion can arise.

We shall now treat each of the terms in (5.10) separately only pointing out the relevant informations for bounding them appropriately.

Beginning with the $\chi\tilde{\chi}$ term, performing an integration by parts (IP), one gets

$$\begin{aligned}&\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)\chi(\xi)\tilde{\chi}(r^2\xi)\xi^{-1/2}d\xi = \\ &\frac{1}{t}\frac{1}{r}\int_0^{\epsilon_2} sin(t\xi^{1/2})\partial_\xi\left(v_0(\xi)\phi(r,\xi)\chi(\xi)\tilde{\chi}(r^2\xi)\right)d\xi.\end{aligned} \tag{5.11}$$

The $\tilde{\chi}(r^2\xi)$ is forcing the regime $r^2\xi \lesssim 1$ or equivalently $r \lesssim \xi^{-1/2}$.
Because ξ is ranging in $[0,\epsilon_2]$ during the integration process while r can be considered to be fixed, one conludes that r is bounded from above by $\epsilon_2^{-1/2}$.
Therefore our main concern when bounding $\|.\|_\infty$ will be to control the absolute value of the right-hand side of (5.11) as $r \to 0$. In other words one shall have to carefully cancel occurrences of negative powers of r.

Considering the Taylor series expression for ϕ given in Proposition 4.1 and its ξ-derivative given by (5.9) one is allowed to write them in compact form as $\phi_\xi = \phi = O(r)$ as $r \to 0$, essentially using (4.9) and (4.10).

Summing up those observations one concludes that he will have control of $\|.\|_\infty$ of the $\chi\tilde{\chi}$-term in $O(t^{-1})$ by imposing finiteness of

$$\int_0^{\epsilon_2} |v_0| d\xi \tag{5.12}$$

and

$$\int_0^{\epsilon_2} |v_0^{'}| d\xi\ . \tag{5.13}$$

Note that all the cutoffs could have been bounded by 1 before performing IP.
In other words the introduced cutoff functions will have no impact on bounding issue as it has to be.

For treating the term (5.12) one writes it as

$$\int_0^{\epsilon_2} |v_0| d\xi = \int_0^{\epsilon_2} |v_0|\xi^{1/2}\xi^{-1/2}d\xi$$

revealing the spectral density.
By the Cauchy-Schwartz inequality the right-hand side of the latter is controlled in $O(\|v_0\|_{L^2(\mathbb{R}_+,d\rho)})$ because $\xi^{1/2} \in L^2([0,\epsilon_2),\xi^{-1/2}d\xi)$ and

$$v_0 \in L^2(\mathbb{R}_+, d\rho)\ (\subset L^2([0,\epsilon_2),\xi^{-1/2}d\xi) \text{ when considering restriction}).$$

This last property is resulting from the fact that v_0 is a Fourier transform if one impose

$r u_0 \in L^2(\mathbb{R}_+, dr)$.

To obtain control of (5.13) one imposes the stronger condition

$$r < r > u_0 \in L^2(\mathbb{R}_+, dr), \tag{5.14}$$

which is sufficient to control simultaneously, by standard Fourier transform properties, the $L^2(\mathbb{R}_+, d\rho)$ norm of $v_0(\xi)$ and $v_0^{'}(\xi)$.

We now turn to the $\chi(1-\tilde{\chi})$ term, that is the second integral term in (5.10).

We shall use the Jost representation (4.44) for ϕ with $a(\xi) \asymp 1$, remembering that a has symbolic behaviour under the derivative operations.
The range for r here, again considering $\|.\|_\infty$ bounding, is as $r \to \infty$.

For ϕ_ξ one has

$$\begin{aligned}
&\phi_\xi \approx \partial_\xi \left(a(\xi) e^{ir\xi^{1/2}} \sigma(q,r) \right) \approx \\
&\frac{1}{\xi} e^{ir\xi^{1/2}} \sigma(q,r) + e^{ir\xi^{1/2}} r \xi^{-1/2} \sigma(q,r) + e^{ir\xi^{1/2}} \sigma(q,r)_\xi \approx \\
&\frac{1}{\xi} e^{ir\xi^{1/2}} \sigma(q,r) + e^{ir\xi^{1/2}} r \xi^{-1/2} \sigma(q,r) + e^{ir\xi^{1/2}} \frac{1}{\xi} \xi \partial_\xi \sigma(q,r)\,,
\end{aligned} \tag{5.15}$$

having omitted some irrelevant constants.

Performing IP in the second term of (5.10), the only new condition on u_0 could come from the term involving $v_0 \phi_\xi$ in the IP procedure. This is due to the fact that $\phi = O(1)$ as $\xi \in (0, \epsilon_2)$ and $r \to \infty$. This last fact is easily seen by writing

$$\phi = a(\xi) e^{ir\xi^{1/2}} \left(\sigma(q,r) - 1 \right) + a(\xi) e^{ir\xi^{1/2}},$$

and taking into account the asymptotic sum behaviour for σ given by (4.13) for being able to bound $\sigma(q,r) - 1$ in $O(\frac{1}{qr^2}) = O(1)$ when $r \to \infty$.

We therefore treat the contributions coming from the three terms on the third line of (5.15).

The first one will lead to the control of

$$\int_0^{\epsilon_2} |v_0| \xi^{-1/2} d\xi,$$

the other $\xi^{-1/2}$ removed by observing that $\frac{1}{r\xi^{1/2}} = O(1)$ considering our regime (the range for the q-variable) due to the cutoff $1-\tilde{\chi}$.
We therefore have no additional condition on u_0 other than the one already stated in (5.14).

The contribution coming from the second term in the third line of (5.15) is easily controlled in $\|.\|_\infty$-norm only pointing out that the r in $e^{ir\xi^{1/2}} r\xi^{-1/2}\sigma(q,r)$ is removing the $1/r$ before inegral in (5.10).

The contribution resulting from the third term in the third line of (5.15) implies no additional condition on u_0 too.
To see this one essentially uses the fact that the action of the operator $\xi\partial_\xi$ on σ is the same as that of $q\partial_q$ as one can write $\xi\partial_\xi\sigma(r\xi^{1/2},r) = \xi\partial_q\sigma(r\xi^{1/2},r)r\xi^{-1/2} = q\partial_q\sigma(q,r)$, using that $q = r\xi^{1/2}$.
Therefore, by (4.13), one has control

$$|e^{ir\xi^{1/2}}\frac{1}{\xi}\xi\partial_\xi\sigma(q,r)| \lesssim \xi^{-1},$$

and can absorb $\xi^{-1/2}$ again using our regime $r\xi^{1/2} > 1$.

The $\chi(1-\tilde{\chi})$ term in (5.10) is therefore controlled in $\|.\|_\infty$-norm in $O(t^{-1})$ without any further assumptions on u_0 than the one stated in (5.14).

We now concentrate on the $(1-\chi)\tilde{\chi}$ term given by

$$\frac{1}{r}\int_0^\infty cos(t\xi^{1/2})v_0(\xi)\phi(r,\xi)(1-\chi)\tilde{\chi}\xi^{1/2}d\xi.$$

After having inserted $\xi^{1/2}\xi^{-1/2}$ and performed an IP one obtains

$$\begin{aligned}&\frac{1}{t}\frac{1}{r}\int_{\epsilon_1}^\infty sin(t\xi^{1/2})\big(v_0(\xi)\phi(r,\xi)(1-\chi)\tilde{\chi}\xi\big)_\xi\, d\xi + \\ &\text{boundary term at infinity}.\end{aligned} \tag{5.16}$$

The boundary term at infinity is given by the limit of

$$\frac{1}{r}sin(t\xi^{1/2})v_0(\xi)\phi(r,\xi)(1-\chi)\tilde{\chi}\xi \tag{5.17}$$

as $\xi \to \infty$ for fixed $r \in (0, \epsilon_1^{-1/2})$.
Using the cutoff $\tilde{\chi}$ this limit is 0 bringing therefore no boundary term issue.

Taking advantage of the behaviour of ϕ in the regime $q \lesssim 1$ and for the range $r \to 0$, the term resulting from IP requiring the strongest condition on u_0 is the one expressed by

$$\int_{\epsilon_1}^{\infty} v_0^{'}(\xi)\xi d\xi = \int_{\epsilon_1}^{\infty} v_0^{'}(\xi)\xi^{1/2}\xi^{1/2} d\xi\,, \tag{5.18}$$

highlighting the spectral density.
For obtaining a convergent integral on the right-hand side of (5.18) one expresses the latter as

$$\int_{\epsilon_1}^{\infty} v_0^{'}(\xi)\xi^{1/2}\xi^{b}\xi^{-b}\xi^{1/2} d\xi\,, \tag{5.19}$$

finds $b > 0$ such that $\xi^{-b} \in L^2([\epsilon_1,\infty),\xi^{1/2}d\xi)$ and finally asks for $v_0^{'}(\xi)\xi^{1/2}\xi^{b} \in L^2(\mathbb{R}_+,\rho)$.
One finds that b has to be in $(3/4,\infty)$.

One concludes to a new requirement for u_0 as

$$< r > r u_0 \in H^{5/2(+)}(\mathbb{R}_+, dr). \tag{5.20}$$

At this point a remark is in order to argue about the Sobolev regularity in (5.20).

The power in ξ multiplying $v_0^{'}$ resulting from the requirements made just after (5.19) is $\frac{1}{2} + 3/4(+) = 5/4(+)$. One then asks for $v_0^{'}(\xi)\xi^{5/4(+)} \in L^2(\mathbb{R}_+,\rho)$ and in the usual Fourier transform setting this would be satisfied by imposing $< r > r u_0 \in H^{5/4(+)}(\mathbb{R}_+, dr)$.

A heuristic argument for the doubling in the Sobolev regularity in the present context of the Distrted Fourier Transform, given in (5.20), is the fact that multiplying by ξ the Fourier Transform $\hat{u}$ of an $L^2(\mathbb{R}_+, dr)$ function u corresponds, by Theorem 3.23, to the application of the operator $\mathscr{L} = -\frac{d^2}{dr^2} + V(r)$ on u.
Taking into account that $V \approx < r >^{-4}$ and all of its derivatives are bounded functions in r implies therefore an L^2 control of the order of derivatives of u two times the related order of the polynomial multiplying the Fourier Transform $\hat{u}$.

A complete argument would require interpolation theory to treat the case of non-integer power of ξ and can be found in [17].

An other related indicator leading to the same conclusion is the fact that the Fourier basis ϕ, when expressed in the Jost representation form, is given by an expression where the free wave factor $e^{ir\xi^{1/2}}$ is present.

One is left with considering the last integral term in (5.10), namely the one with the cutoff form $(1-\chi)(1-\tilde{\chi})$.

The only relevant additional information compared to what has been developed from the beginning of the proof is the following formula giving the terms emanating from the ξ derivative of ϕ in the Jost representation form when $\xi \to \infty$

$$\begin{aligned} &\phi_\xi \approx \partial_\xi \left(a(\xi) e^{ir\xi^{1/2}} \sigma(q,r) \right) \approx \\ &\xi^{-3/2} e^{ir\xi^{1/2}} \sigma(q,r) + e^{ir\xi^{1/2}} r\xi^{-1} \sigma(q,r) + e^{ir\xi^{1/2}} \sigma(q,r)_\xi \approx \\ &\xi^{-3/2} e^{ir\xi^{1/2}} \sigma(q,r) + e^{ir\xi^{1/2}} r\xi^{-1} \sigma(q,r) + e^{ir\xi^{1/2}} \frac{1}{\xi} \xi \partial_\xi \sigma(q,r) \,, \end{aligned} \tag{5.21}$$

as $|a(\xi)| \asymp \xi^{-1/2}$ with symbolic form for derivatives. Again signs and possible constants are irrelevant in this context.

One concludes that the control of the $\|.\|_\infty$-norm of the $(1-\chi)(1-\tilde{\chi})$ term requires no additional conditions on u_0 essentially because all the terms in the last line of (5.21) include negative power in ξ resulting in less Sobolev regularity for u_0 when considering the convergence of integrals similar to that in (5.18).

Summing up the discussion so far one obtains the following linear dispersive estimate for $A(t)$ as

$$\begin{aligned} \|A(t)u_0\|_\infty \leq \tilde{C}\frac{1}{t} \|r < r > u_0\|_{H^{5/2(+)}(\mathbb{R}_+, dr)} = \\ C\frac{1}{t} \| < r > u_0\|_{H^{5/2(+)}(\mathbb{R}^3, dx)}. \end{aligned} \tag{5.22}$$

This completes the proof of Lemma 5.1. □

Sinus Evolution

In this section we are focussing on the operator $B(t)$ defined by (5.2).
Our goal would be to obtain, in the same spirit as was done for the cosinus evolution operator, some linear dispersive estimates for it.

If $u_1 \in P_c\mathscr{D}(H)$, one write its action as

$$B(t)u_1 = \frac{1}{r} \int_0^\infty \frac{sin(t\xi^{1/2})}{\xi^{1/2}} v_1(\xi) \phi(r,\xi) \rho(\xi) d\xi \tag{5.23}$$

where, as above, $v_1(\xi) = \widehat{r u_1}(\xi)$, that is the Fourier transform of $r u_1$.

We shall see that this additional singular term in $\xi^{-1/2}$ will cause problem for controlling the action of $B(t)$.
The price to pay for obtaining linear dispersive estimates from the action of $B(t)$ will be to extract from the latter a resonance part (similar to a trace class operator but using resonance) revealing no time decay.

We shall prove the following

Lemma 5.3. *$B(t)$ can be expressed as*

$$B(t) = c_0 \frac{\phi_0}{r} < . , \frac{\phi_0}{r} > + S(t), \tag{5.24}$$

where $c_0 \in \mathbb{R}$ is a non-zero universal constant.

Moreover $S(t)$ is an operator satisfying the following linear dipersive estimate

$$\| S(t) u_1 \|_\infty \leq \tilde{C} \frac{1}{t} \| < r > u_1 \|_{H^{3/2(+)}(\mathbb{R}^3, dx)}, \tag{5.25}$$

with $\tilde{C} > 0$ a universal constant (not depending on u_1 and $t > 0$)

Remark 5.4. The scalar product notation in the above formula for $B(t)$ can be misleading because the resonance $\frac{\phi_0}{r}$ is not in $L^2(\mathbb{R}^3)$.
It is nevertheless defined by the same expression as if it was an L^2 scalar product essentially by imposing $r u_1 \in L^2(\mathbb{R}_+, dr)$ (or equivalently $u_1 \in L^2_{rad}(\mathbb{R}^3, dx)$) and using the Distorted Fourier Transform representation. See below for the details.

Proof. We begin by breaking, as was done for the cosinus propagator, the integral part in (5.23) in the two range as $\xi \to 0$ and $\xi \to \infty$ as

$$\begin{aligned} &\frac{1}{r} \int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} \chi(\xi) v_1(\xi) \phi(r, \xi) \xi^{-1/2} d\xi + \\ &\frac{1}{r} \int_{\epsilon_2}^{\infty} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} (1 - \chi(\xi)) v_1(\xi) \phi(r, \xi) \xi^{1/2} d\xi \, . \end{aligned} \tag{5.26}$$

The issue for obtaining linear dispersive estimates coming from the first integral in (5.26), we express it as

$$\begin{aligned}
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)v_1(\xi)\phi(r,\xi)\xi^{-1/2}d\xi = \\
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)v_1(0)\phi(r,\xi)\xi^{-1/2}d\xi + \\
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)(v_1(\xi)-v_1(0))\phi(r,\xi)\xi^{-1/2}d\xi.
\end{aligned} \tag{5.27}$$

A remark is in order concerning the two terms on the right-hand side of (5.27).

The first one containing $v_1(0)$ will certainly not be tractable under the form of a linear dispersive estimate because of too much singularity in ξ.
The second looks better in order to obtain some linear dispersive estimate because one will be able to use the Mean Value Theorem for removing part of the singularity in ξ.

The first term on the right-hand side of (5.27) can be expressed (modulo some universal constant) as

$$\begin{aligned}
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)v_1(0)\phi(r,\xi)\xi^{-1/2}d\xi = \\
&< u_1, \frac{\phi_0}{r} > \frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\phi(r,\xi)\xi^{-1/2}d\xi,
\end{aligned} \tag{5.28}$$

having expressed $v_1(0)$ using Fourier transform as

$$v_1(0) = \int_0^{\infty} r u_1 \phi(r,0)dr = \int_0^{\infty} r u_1\phi_0(r)dr = \int_0^{\infty}\frac{\phi_0}{r}u_1 r^2 dr \approx \int_{\mathbb{R}^3}\frac{\phi_0}{r}u_1 dx, \tag{5.29}$$

where the last integral is not strictly an $L^2(\mathbb{R}^3)$ scalar product as it should have been suggested by the expression on the right-hand side of (5.28). In fact the resonance in dimension 3, ϕ_0/r, is not belonging to $L^2(\mathbb{R}^3)$.

It thus remains to analyze the function multiplying $< u_1, \frac{\phi_0}{r} >$ on the right-hand side of (5.28) namely

$$\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\phi(r,\xi)\xi^{-1/2}d\xi. \tag{5.30}$$

For being able to develop further one extracts the resonance term from (5.30) writing

$$\begin{aligned}
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\phi(r,\xi)\xi^{-1/2}d\xi = \\
&\frac{\phi_0(r)}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\xi^{-1/2}d\xi + \\
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\tilde{\chi}(r^2\xi)(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi + \\
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)(1-\tilde{\chi}(r^2\xi))(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi,
\end{aligned} \tag{5.31}$$

where the same types of cutoffs than those used in (5.10) appear.

The second term on the right-hand side of (5.31) is developed, as was done in the cosinus case, using the Taylor series expression for ϕ under the following form

$$\phi(r,\xi) = \phi_0(r) + O(r^2\xi),$$

taking advantage of the regime $r^2\xi \lesssim 1$.
This results in

$$\begin{aligned}
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\tilde{\chi}(r^2\xi)(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi \approx \\
&\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\tilde{\chi}(r^2\xi)O(r^2\xi)\xi^{-1/2}d\xi.
\end{aligned} \tag{5.32}$$

The symbolic expression $O(r^2\xi)$ enables one to eliminate some singular part in ξ by writing

$$\frac{1}{r}O(r^2\xi) = O(r\xi) = O(\xi^{1/2}) \tag{5.33}$$

the first $1/r$ being the one in front of integrals in (5.32). In the second equality one takes advantage of the regime $r^2\xi \le 1$.
This reduces the right-hand side of (5.32) to the following

$$\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\tilde{\chi}(r^2\xi)d\xi, \tag{5.34}$$

the latter being trivially bounded in $O(t^{-1})$, uniformly in r, by an IP considering the finiteness of the measure $\xi^{-1/2}d\xi$ on $(0,\epsilon_2)$.

The third term on the right-hand side of (5.31) namely

$$\frac{1}{r}\int_0^{\epsilon_2}\frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)(1-\tilde{\chi}(r^2\xi))(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi$$

is also bounded in L^∞ norm.
To see this one has first to eliminate one $\xi^{-1/2}$ singularity taking advantage of the regime $1 \lesssim r\xi^{1/2}$. In a second time one will use that $\phi(r,\xi) - \phi_0$ is $O(1)$ as $r \to \infty$ essentially using (4.9) and the asymptotic sum representation for σ given by (4.13) in the Jost form for ϕ.
Finally one is left with $\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} \chi(\xi) d\xi$ which is again bounded in $O(t^{-1})$ using IP, the cutoff χ enabling to eliminate the boundary term at ϵ_2.

We now take care of the first term on the right-hand side of (5.31).
We shall use the fact that $sin(x)/x$ is continuous on $[0,1]$, being therefore integrable there, and the oscillating character of the integrand enabling one to get $O(t^{-1})$ decay for the non-constant (in t) part of the expression. Details are as follows.

We define c_0 by

$$c_0 := \int_0^\infty \frac{sin(u)}{u} du. \tag{5.35}$$

Performing the change of variable $u = t\xi^{1/2}$ in the first term on the right-hand side of (5.31) multiplying $\frac{\phi_0}{r}$ one obtains

$$\int_0^{\epsilon_2^{1/2} t} \frac{sin(u)}{u} \chi(\frac{u^2}{t^2}) du, \tag{5.36}$$

with the boundary value depending linearly on t.

One breaks (5.36) as

$$\int_0^{\epsilon_2^{1/2} t} \frac{sin(u)}{u} du + \int_0^{\epsilon_2^{1/2} t} \frac{sin(u)}{u} (\chi(\frac{u^2}{t^2}) - 1) du. \tag{5.37}$$

The first integral in (5.37) is then expressed as

$$\int_0^\infty \frac{sin(u)}{u} du - \int_{\epsilon_2^{1/2} t}^\infty \frac{sin(u)}{u} du, \tag{5.38}$$

the second term in (5.38) being easily controlled in $O(t^{-1})$ by considering an IP and the boundedness of the sin function.
Note that $\frac{1}{u}$ is not integrable on (a,∞) with $a > 0$ but $\frac{sin(u)}{u}$ is integrable due to the oscillatory character of the integrand.

Now the second term in (5.37) is also controlled in $O(t^{-1})$ because the domain of integration is $(\epsilon_1^{1/2} t, \epsilon_2^{1/2} t)$ due to the cutoff χ. One can therefore bound the absolute value of $\int_{\epsilon_1^{1/2} t}^{\epsilon_2^{1/2} t} \frac{sin(u)}{u} (\chi(\frac{u^2}{t^2}) - 1) du$ by $\int_{\epsilon_2^{1/2} t}^\infty |\frac{sin(u)}{u}| du$ the latter being $O(t^{-1})$ by the same consideration as above.

For ending up with the treatment of the χ term, we are left with the second integral term on the right-hand side of (5.27).

We express the latter using the Mean Value Theorem as

$$\frac{1}{r}\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} v_1^{'}(\tau(\xi))\phi(r,\xi)\chi(\xi)\xi^{1/2} d\xi, \tag{5.39}$$

where $\tau(\xi) \in (0,\xi)$ such that $v_1(\xi) - v_1(0) = v_1^{'}(\tau(\xi))\xi$.
One observes, by basic Fourier theory, that $v_1^{'}$ will exist almost everywhere and will even be $L^1(\mathbb{R}_+, d\rho)$ if one asks for $r < r > u_1 \in H^{3/2(+)}(\mathbb{R}_+, dr)$. Because then v_1 and $v_1^{'}$ are then L^1 functions basic Sobolev theory gives that v_1 is absolutely continuous (and therefore continuous). One also remarks that the singularity coming from the spectral density was removed by the ξ coming from the Mean Value theorem.

As usual, for being able to bound the $\|.\|_\infty$ norm of this term, we break it using a second cutoff $\tilde{\chi}$ writing

$$\begin{aligned} &\frac{1}{r}\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} v_1^{'}(\tau(\xi))\phi(r,\xi)\chi(\xi)\xi^{1/2} d\xi = \\ &\frac{1}{r}\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} v_1^{'}(\tau(\xi))\phi(r,\xi)\chi(\xi)\tilde{\chi}(r^2\xi)\xi^{1/2} d\xi + \\ &\frac{1}{r}\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} v_1^{'}(\tau(\xi))\phi(r,\xi)\chi(\xi)(1-\tilde{\chi}(r^2\xi))\xi^{1/2} d\xi. \end{aligned} \tag{5.40}$$

Again the r dependance of the two terms on the right-hand side of (5.40), when considering $\|.\|_\infty$ norm bounding, can be controlled using either of the two representations for the Fourier basis and the range for the regime variable $r\xi^{1/2}$.

We are thus left to deal with the following integral (having also dropped out the cutoffs, bounded by 1)

$$\int_0^{\epsilon_2} sin(t\xi^{1/2}) v_1^{'}(\tau(\xi))\chi(\xi)\xi^{1/2}\xi^{-1/2} d\xi. \tag{5.41}$$

Because of the presence of $v_1^{'}$ we shall have to isolate it before performing IP for avoiding the control of second derivative of v_1.

The first step is to show that

$$v_1^{'}(\tau(\xi))$$

is continuous on $[0, \epsilon_2)$.

On the one hand one has continuity of $v_1'(\tau(\xi))$ for $\xi \in (0, \epsilon_2)$ by writing

$$v_1'(\tau(\xi)) = \frac{v_1(\xi) - v_1(0)}{\xi}. \tag{5.42}$$

On the other hand considering limit as $\xi \to 0$ of the right-hand side of (5.42) gives well-defined limit as $v_1'(0)$. $v_1'(\tau(\xi))$ is then continuous on $[0, \epsilon_2)$.

We finally bound (5.41) using the Cauchy-Schwartz inequality as

$$\begin{gathered} |\int_0^{\epsilon_2} sin(t\xi^{1/2})\chi(\xi)v_1'(\tau(\xi))\xi^{1/2}\xi^{-1/2}d\xi| \lesssim \\ \|v_1'\|_{L^2(\mathbb{R}_+,d\rho)} \left(\int_0^{\epsilon_2} sin(t\xi^{1/2})^2\chi(\xi)^2\xi\xi^{-1/2}d\xi\right)^{1/2}, \end{gathered} \tag{5.43}$$

having used that

$$r < r > u_1 \in L^2(\mathbb{R}_+, dr).$$

To obtain a bound of (5.43) in $O(t^{-1})$, due to the square root on the right-hand side of (5.43), we have to perform two integrations by parts in the integral of the right-hand side of (5.43).

Performing a first IP for the integral inside square root in (5.43) results in terms as

$$\begin{gathered} \frac{1}{t}\int_0^{\epsilon_2} cos(t\xi^{1/2})\left(\chi(\xi)\chi(\xi)\xi\right)_\xi d\xi \approx \\ \frac{1}{t}\int_0^{\epsilon_2} cos(t\xi^{1/2})\chi(\xi)'\chi(\xi)\xi d\xi + \frac{1}{t}\int_0^{\epsilon_2} cos(t\xi^{1/2})\chi(\xi)^2 d\xi, \end{gathered} \tag{5.44}$$

where we bounded one of the sinus in the integral under square root in (5.43) by 1 before performing IP.
Note that boundary terms at ϵ_2 and 0 vanish respectively because of the presence of the cutoff χ and the ξ inside the integral on the left-hand side of (5.44).

Now a second IP for each of the two terms on the right-hand side of (5.44) can be performed without any difficulty only pointing out that again boundary terms at 0 are vanishing due to the presence of sin function.

It only remains to work on the $1-\chi$ part of the action of $B(t)$ on u_1.

In the present case, as was already observed for the cosinus evolution, there is no problem performing IP principally because one can always add some negative power of ξ (and its positive counterpart associated with v_1) asking for a multiplication of two L^2 functions. We shall therefore not give the easy details of calculation only stating the following result.

The final condition on u_1, found by controlling the $\|.\|_\infty$ norm of $1-\chi$ terms in $O(t^{-1})$, is given by

$$r < r > u_1 \in H^{3/2(+)}(\mathbb{R}_+, dr). \tag{5.45}$$

Summing up all the discussion so far we have succeeded in obtaining a decomposition of the operator $B(t)$ in

$$B(t) = c_0 \frac{\phi_0}{r} < . , \frac{\phi_0}{r} > + S(t),$$

where the action of $S(t)$ is controlled in $\|.\|_\infty$-norm in $O(t^{-1})$.

To get the linear dispersive estimate

$$\|S(t)u_1\|_\infty \leq \tilde{C}\frac{1}{t}\| < r > u_1\|_{H^{3/2(+)}(\mathbb{R}^3, dx)}$$

one only remarks that $< u_1, \frac{\phi_0}{r} >$ can be bounded in absolute value using the Cauchy-Schwartz inequality thanks to the condition on u_1 given in (5.45).
More precisely one writes

$$< u_1, \frac{\phi_0}{r} > \approx \int_{\mathbb{R}^3} \frac{\phi_0}{r} u_1 dx = \int_{\mathbb{R}^3} \frac{\phi_0}{< r > r} < r > u_1 dx, \tag{5.46}$$

the integrands $\frac{\phi_0}{<r>r}$ and $< r > u_1$ being then both in $L^2(\mathbb{R}^3, dx)$.

This concludes the proof of the lemma.

□

Remark 5.5. An interesting point to note about the expression for the sinus propagator given in Lemma (5.3) is the fact that the action of the linear dispersive controlled part $S(t)$ do not reveal a tractable Fourier transform.
More precisely one can not write the Fourier transform of $rS(t)u_1$ essentially because the non-resonance terms on the right-hand side of (5.31) are expressed using $\phi(r,\xi) - \phi_0$ in place of the Fourier basis itself.

6 Stable Manifold

In this chapter we shall construct a stable manifold of Cauchy values for radial solutions of (1.1), or more precisely for radial waves which are radiations around some static Aubin-Talenti solution given in (1.3).

This consists of a subset of the phase space composed of starting values for the radiative part of the wave $(u_t, \partial_t u)$ at $t = 0$ such that it exists globally in time when $t \to \infty$.
It shall further have the property to make the radiative part u scatters to a free wave as $t \to \infty$.

Introduction

We are looking for a solution of (1.1) of the form $\phi(., a_\infty) + u(r, t)$ with

$$\{\phi(., a)\ ,\ a > 0\}$$

being the static Aubin-Talenti solutions given in (1.3). $a_\infty > 0$ is a parameter value eventually being the limit of a sequence $(a_n)_n \in \mathbb{R}_+$ of approximative values obtained in a recursive scheme needed to construct the scattering solution u.

For the confort of reading one recalls the theorem we are going to prove in this chapter.

Theorem 6.1. *Fix* $0 < \delta < 1$ *and let*

$$\begin{aligned} \mathscr{B}_{1,\delta} &:= \{f \in L^2(\mathbb{R}^3)\ :\ \| < r > f\|_{H^{5/2(+)}(\mathbb{R}^3)} < \delta\} \\ \mathscr{B}_{2,\delta} &:= \{g \in L^2\ :\ \| < r > g\|_{H^{3/2(+)}(\mathbb{R}^3)} < \delta\}. \end{aligned} \tag{6.1}$$

Define then

$$\Sigma := \{(f_0, u_1) \in \mathscr{B}_{1,\delta} \times \mathscr{B}_{2,\delta}\ :< g_1, k_1 f_0 + u_1 >= 0\}. \tag{6.2}$$

Then there exists a Lipschitz function $h:\Sigma\to\mathbb{R}$ with the property that for every pair $(f_0,u_1)\in\Sigma$ one is able to find a positive real number $a_\infty(f_0,u_1)\in(1-\delta,1+\delta)$ such that the Cauchy problem

$$\begin{cases}\Box\psi-\psi^5=0\\ \psi(0,.)=\phi(.,1)+f_0+h(f_0,u_1)g_1\ ,\ \partial_t\psi(0,.)=u_1\end{cases}\tag{6.3}$$

has a unique radial global in positive time solution under the form
$\phi(.,a_\infty)+u(r,t)$ whith u scattering to a free wave in the phase space $\dot{H}^1\times L^2$ when $t\to\infty$.

Remark 6.2. One obtains the starting value $\psi(0,.)=\phi(.,1)+f_0+h(f_0,u_1)g_1$ because the two $\phi(.,a_\infty)$, the one coming from the linearization and the one coming from the correction needed for removing the resonance term at each step of the iterative procedure (see Section 6.3 and (6.153)), are cancelling each other.
This gives the advantage to obtain expression for starting values $\psi(0,.)$ completely independant of a_∞ which is clearly depending on the Cauchy values (f_0,u_1) in an implicit way (see (6.148)).

We shall break the proof of Theorem 6.1 in different subsections where we prove intermediate useful results. This will have the advantage to simplify the reading eventually giving the proof of the scattering to free wave as the last part of the proof of Theorem 6.1 in a relatively short section.

The Linearized Problem

We start by considering which are the different requirements for a solution of

$$\begin{aligned}&\partial_{tt}u+H(a_\infty)u=N(u,\phi_\infty)\\&(u(0),\partial_t u(0))=(u_0,u_1),\end{aligned}\tag{6.4}$$

with Cauchy starting values (u_0,u_1), to exist and to scatter to free-wave as $t\to\infty$.
This shall bring a list of equations on which the iterative procedure will be constructed.

Define $H(a(\infty))=:H_\infty$.

By properties of the resonance and using Agmon estimates, one proves that, in radial case (that is working with radial functions), H_∞ has only one negative eigenvalue noted $-k_\infty^2$ with $k_\infty>0$ of multiplicity one. The associated ground state g_∞, noramlized as $\|g_\infty\|_2=1$, is such that $g_\infty>0$ and has exponential decay.

The ground state of H_∞ will lead to some instability for the wave.
To get a better understanding of this dynamics, essentially for being able to control the evo-

lution of the wave in discrete spectrum, the easiest way is to pass to ODE notation and then projecting. This dynamical system way of writing the wave equation will then enable us to obtain a tracatble first order (in time) system of ODE governing the discrete spectrum evolution. The price to pay will be the analysis of a new matrix operator.

Rewriting (6.4) as an ODE system in $L^2_{rad} \times L^2_{rad}$ gives the new form as

$$\dot{U} = J\mathscr{H}_\infty U + W,\ U(0) = U_0, \tag{6.5}$$

where $U = (u, \partial_t u)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathscr{H}_\infty = \begin{pmatrix} H_\infty & 0 \\ 0 & 1 \end{pmatrix}$, and the starting values written as $U_0 = (u_0, u_1)$. Moreover $W = \begin{pmatrix} 0 \\ N(u, \phi_\infty) \end{pmatrix}$ is the nonlinearity.

If one wants the explicit dependance on t of U he shall write

$$U(t) = (u(t,.), \partial_t u(t,.)).$$

A very important observation relevant at this point is that one does not know right now what is exactly the phase space for the wave U.
What can be said at the moment is that $u(t,.)$ lies certainly not in $L^2(\mathbb{R}^3)$. To see this just observe the form for the sinus propagator obtained in Lemma 5.3 noting then that in the expression for u there will be a resonance part which is not in L^2 (see below (6.51)).
It is therefore highly probable that $\dot{H}^1 \times L^2$ will do the job for the phase space corresponding to $U(t,.)$.
One can convince himself by taking into account the propagator for the wave U given in (6.15) which shows that the time derivative of the wave $\partial_t u(t,.)$ is not propagating through an operator whose frequency representation (spectral representation) becomes singular at 0 as is the case with the wave u itself due to $\frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}$.

Nevertheless the framework for investigating the dynamics of the wave U is $L^2_{rad} \times L^2_{rad}$ essentially because the matrix operator $J\mathscr{H}_\infty$ is naturally acting on this space and its spectral properties can be well described in this Hilbert space setting.

Therefore when scalar product notations arises, as is the case when defining the projector $P_\pm$ in (6.9), or when considering the action of the projector on essential spectrum of $J\mathscr{H}_\infty$ which is written in coordinate as the projector orthogonal to the ground state of H_∞ as will be seen in (6.12), one has to be cautious so to make the expressions involving u meaningfull.

Basically all the scalar product expressions seen in the following will be meaningfull when one is asking for the prealable two conditions

$$\begin{cases} u(t,.) \in L^{\infty}(\mathbb{R}^3) \\ P^{\perp}_{g_{a_\infty}} \phi_0^{a_\infty} := \phi_0^{a_\infty}. \end{cases} \tag{6.6}$$

The second condition in (6.6) is the action of the projector on the corresponding resonance and is thus a priori not defined due to the fact that the resonance is not in L^2. One shall therefore think about this latter expression as a definition of the action of the projector $P^{\perp}_{g_\infty}$ on $\phi_0^{a_\infty}$ as if the resonance $\phi_0^{a_\infty}$ was a usual eigenvector of H_∞ which would then be orthogonal to g_∞ due to the self-adjointness of H_∞.

More conditions on u will be imposed as we go along.

The discrete spectrum of $J\mathcal{H}_\infty$ is given by $\{\pm k_\infty\}$ with normalized eigenfunctions as

$$G_\pm = (2k_\infty)^{-1/2} \begin{pmatrix} g_\infty \\ \pm k_\infty g_\infty. \end{pmatrix} \tag{6.7}$$

The need for this normalization will become clear in a moment.

The goal being to project the dynamics on those vectors, we shall need the dual eigenvectors, that is the discrete space for the adjoint of $J\mathcal{H}_\infty$ (mentionning that $J\mathcal{H}_\infty$ is not self-adjoint). We are thus looking for $G^\star_\pm$ such that

$$(J\mathcal{H}_\infty)^\star G^\star_\pm = \pm k_\infty G^\star_\pm, \tag{6.8}$$

observing that k_∞ is real.
We write

$$(\mathcal{H}_\infty)^\star J^\star = -\mathcal{H}_\infty J,$$

giving

$$-\mathcal{H}_\infty J G^\star_\pm = \pm k_\infty G^\star_\pm.$$

Acting by J on both sides results in

$$G^\star_\pm = -J G_\mp.$$

Thanks to the normalization for $G_\pm$ we get

$$< G_\pm, G^\star_\pm >= \pm 1,$$

giving our projectors on discrete spectrum of $J\mathscr{H}_\infty$ as

$$P_\pm = \pm < ., G_\pm^\star > G_\pm. \tag{6.9}$$

We are now in good position to obtain the dynamics on discrete spectrum.

We define $n_\pm(t)$ by

$$n_\pm(t) G_\pm = P_\pm U(t).$$

Differentiating

$$n_\pm(t) = \pm < U(t), G_\pm^\star >$$

in time and considering (6.5) gives the following first order ODE system for $x(t) := (n_+(t), n_-(t))$

$$\dot{x}(t) - \begin{pmatrix} k_\infty & 0 \\ 0 & -k_\infty \end{pmatrix} x(t) = y(t), \tag{6.10}$$

with $y(t) = (F(t), -F(t))$ and

$$F(t) = (2k_\infty)^{-1/2} < N(u, \phi_\infty)(t), g_\infty > . \tag{6.11}$$

Sufficiency for the above scalar product to have a sense is obtained by imposing

$$u(t,.) \in L^\infty.$$

Now if one imposes the stronger condition

$$\| < r > u(t,.)\|_\infty = O(1),$$

this shall ensures that $N(u, \phi_\infty)(t) \in L^2_{rad}$ as it has to be.

Before continuing with discrete spectrum considerations, we discuss what happens for the rest of the spectrum of $J\mathscr{H}_\infty$, that is we consider projection on $\mathscr{H}_e := L^2_{rad} \ominus span\{G_+, G_-\}$ which is the essential spectrum space of $J\mathscr{H}_\infty$.
The projector on essential spectrum of $J\mathscr{H}_\infty$ is given by $P_e := I - P_+ - P_-$.
Written in components, using (6.9), one obtains

$$P_e \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P_{g_\infty}^\perp u \\ P_{g_\infty}^\perp v \end{pmatrix}, \tag{6.12}$$

concluding that projection on essential spectrum of $J\mathscr{H}_\infty$ is the same as projection on essential spectrum of H_∞ in each component. It is also clear, considering (6.12), that the projection P_e is reducing $J\mathscr{H}_\infty$.

If one works on the essential spectrum of $J\mathcal{H}_\infty$, using some computations at the matrix level, he will be able to prove that $J\mathcal{H}_\infty$ is unitarily equivalent to $\begin{pmatrix} i\sqrt{H_\infty} & 0 \\ 0 & -i\sqrt{H_\infty} \end{pmatrix}$, which is well-defined using (6.12). One then concludes that rest of spectrum of $J\mathcal{H}_\infty$ is given by $i\mathbb{R}$.

We now apply P_e to (6.5) obtaining

$$\dot{\tilde{U}} = J\mathcal{H}_\infty \tilde{U} + P_e W,\ \tilde{U}(0) = P_e U_0, \tag{6.13}$$

having written $\tilde{U} := P_e U$. We used commutation of $J\mathcal{H}_\infty$ and P_e being valid by the already mentionned fact that $P_e(L^2_{rad} \times L^2_{rad})$ is reducing $J\mathcal{H}_\infty$.

An important remark to be pointed out is that (6.13) looks similar but is not a Schrodinger evolution equation because of the presence of the symplectic matrix J instead of $i \in \mathbb{C}$.

One has to pass to components for obtaining the propagator.
Writing $\tilde{U} = (\tilde{u}, \partial_t \tilde{u})$, we obtain from (6.13) that

$$\partial_{tt}\tilde{u} + H_\infty \tilde{u} = P^\perp_{g_\infty} N(u, \phi_\infty), \tag{6.14}$$

with boundary values written

$$\tilde{u}(0) = P^\perp_{g_\infty} u_0, \text{ and } \partial_t \tilde{u}(0) = P^\perp_{g_\infty} u_1.$$

Because we are working on $P^\perp_{g_\infty} L^2_{rad} \times P^\perp_{g_\infty} L^2_{rad}$ and because we are in dimension $n = 3$, the propagator takes the following (well-defined) form

$$e^{tJ\mathcal{H}_\infty} := \begin{pmatrix} cos(t\sqrt{H_\infty}) & \frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} \\ -\sqrt{H_\infty} sin(t\sqrt{H_\infty}) & cos(t\sqrt{H_\infty}) \end{pmatrix} \tag{6.15}$$

noting that this is really a definition for $e^{tJ\mathcal{H}_\infty}$ essentially because $J\mathcal{H}_\infty$ is not self-adjoint.

Using this propagator one can write the solution of (6.13) as homogeneous propagation + Duhamel term as

$$\tilde{U} = e^{tJ\mathcal{H}_\infty}\tilde{U}(0) + \int_0^t e^{(t-s)J\mathcal{H}_\infty} P_e W(s) ds. \tag{6.16}$$

We shall need this form for the solution when dealing with scattering issue at the end of the whole argument.

We now come back to the system describing evolution in discrete spectrum (6.10). The decoupled system has solution as

$$\begin{aligned} n_+(t) &= e^{tk_\infty} n_+(0) + \int_0^t e^{(t-s)k_\infty} F(s)\,ds \\ n_-(t) &= e^{-tk_\infty} n_-(0) - \int_0^t e^{-(t-s)k_\infty} F(s)\,ds, \end{aligned} \tag{6.17}$$

observing that evolution n_+ is problematic when looking for globality of solution as $t \to \infty$.

To overcome this difficulty one imposes what is called a *stability condition* by imposing

$$n_+(0) + \int_0^\infty e^{-sk_\infty} F(s)\,ds = 0. \tag{6.18}$$

Taking the condition (6.18) into account, the form for n_+ is given by

$$n_+(t) = -\int_t^\infty e^{(t-s)k_\infty} F(s)\,ds. \tag{6.19}$$

$|n_+(t)|$ is then decaying in time at least as fast than $< t >^{-\beta}$ with $\beta > 0$ if $F(s) \approx < s >^{-\beta}$.

If one imposes

$$\| u(t,.)\|_\infty = O(< t >^{-1}), \tag{6.20}$$

one then gets that $|F(t)| = |(2k_\infty)^{-1/2} < N(u,\phi_\infty)(t), g_\infty > |$
$\leq < t >^{-2}$, by the form of N and the decay property of g_∞.

We have therefore control over the time evolution of n_+.

For $n_-(t)$ given by

$$n_-(t) = e^{-tk_\infty} n_-(0) - \int_0^t e^{-(t-s)k_\infty} F(s)\,ds, \tag{6.21}$$

we break the integral in (6.21) as

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t,$$

to obtain control of $n_-(t)$ in $O(< t >^{-1})$.
We will comment on $n_-(0)$ in a moment.

We shall now use (6.18) to obtain information about the Cauchy data (u_0, u_1). This stability condition coming from the need to control the dynamics on instable mode will provide some condition on Cauchy values which enables global existence (in positive time) of the wave. In particular this will give us the form of the stable manifold for Cauchy data (u_0, u_1).

Writing the solution U as

$$U = n_+ G_+ + n_- G_- + \tilde{U}$$

and considering the first component, one obtains the expression

$$u = (2k_\infty)^{-1/2}(n_+ + n_-)g_\infty + \tilde{u}. \tag{6.22}$$

Performing derivative in time implies

$$\partial_t u = (2k_\infty)^{-1/2}(\dot{n}_+ + \dot{n}_-)g_\infty + \partial_t \tilde{u}. \tag{6.23}$$

Taking value at $t = 0$ and projecting (6.22) and (6.23) on g_∞, one obtains

$$\begin{aligned} n_+(0) + n_-(0) &= (2k_\infty)^{1/2} < u_0, g_\infty > \\ \dot{n}_+(0) + \dot{n}_-(0) &= (2k_\infty)^{1/2} < u_1, g_\infty > . \end{aligned} \tag{6.24}$$

Moreover one deduces from (6.10) that

$$\dot{n}_+(0) + \dot{n}_-(0) = k_\infty(n_+(0) - n_-(0)). \tag{6.25}$$

Taking into account (6.24), (6.25) and (6.18), one gets

$$\begin{aligned} &2(2k_\infty)^{-1/2} < g_\infty, k_\infty u_0 + u_1 >= - \\ &(2k_\infty)^{-1/2} \int_0^\infty e^{-tk_\infty} < N(u, \phi_\infty), g_\infty > dt. \end{aligned} \tag{6.26}$$

On the left-hand side of (6.26) the scalar product notation stands for the effective L^2 scalar product by imposing that $(u_0, u_1) \in L^2_{rad} \times L^2_{rad}$.

By imposing the further condition on u

$$\| u(t,.)\|_\infty = O(\delta < t >^{-1}), \tag{6.27}$$

with $0 < \delta < 1$, the right-hand side of (6.26) is $O(\delta^2)$ giving scaling to the equality (6.26).
For the moment the value of δ is not further specified. We shall comment about further conditions on its value if needed.

We define the following

$$a_{-1} := 1, \tag{6.28}$$

which will basically represent the (arbitrary) starting value for the sequence of positive parameters where linearization around the associated Aubin-Talenti solution will be iteratively performed.

Introducing f_0 by expressing u_0 as

$$u_0 = f_0 + h(f_0, u_1) g_{a_{-1}}, \tag{6.29}$$

where $h(f_0, u_1) \in \mathbb{C}$ and $g_{a_{-1}}$ being the ground state associated to linearization around $\phi(., a_{-1})$. One has to understand that, in expressing u_0 under the form (6.29), we are implicitely defining $f_0 \in L^2_{rad}$ and $h \in \mathbb{C}$ satisfying (under the form) $f_0 = u_0 - h g_{a_{-1}}$ and the goal is to show that the stability condition (6.18) enables one to determine (f_0, h) uniquely. Moreover h will happen to be a function of (f_0, u_1, u, a_∞) which is Lipschitz in the variables (f_0, u_1).
The need for the use of $g_{a_{-1}}$ and the specific notations will become clear shortly.

Remember that the expressions g_a and k_a stand for the ground state and associated square root of the absolute value of the eigenvalue retated to the linearization around the Aubin-Talenti solution with parameter $a > 0$, and are given in (1.10).

For being able to succeed in showing that h is well-defined, one will have to impose some conditions on the Cauchy values (f_0, u_1) which will then restrict them to some codimension one subset of the phase space.
This shall represent the parameter space for the stable manifold.

By inserting (6.29) in (6.26), one obtains

$$\begin{aligned} &< g_\infty, k_\infty f_0 + u_1 > + k_\infty h(f_0, u_1) < g_{a_{-1}}, g_\infty >= \\ &-\frac{1}{2}\int_0^\infty e^{-tk_\infty} < N(u, \phi_\infty), g_\infty > dt. \end{aligned} \tag{6.30}$$

The right-hand side being $O(\delta^2)$, it has therefore to be the same for the left-hand side.

To get an $O(\delta^2)$ scale for the left-hand side of (6.30) one writes for the first term there

$$\begin{aligned} &< g_\infty, k_\infty f_0 + u_1 >=< g_\infty - g_{a_{-1}}, k_{a_{-1}} f_0 + u_1 > + \\ &< g_\infty - g_{a_{-1}}, (k_\infty - k_{a_{-1}}) f_0 > + < g_{a_{-1}}, (k_\infty - k_{a_{-1}}) f_0 >, \end{aligned} \tag{6.31}$$

where we imposed

$$< g_{a_{-1}}, k_{a_{-1}} f_0 + u_1 >= 0. \tag{6.32}$$

This last equation represents the codimension one condition on (f_0, u_1).

We had to impose condition (6.32) because it is the only term appearing on the right-hand side of (6.31) which we have no possibility to write as $O(\delta^2)$.

To be more specific assume that we have a norm scale for $(u_0, u_1) \in L^2_{rad} \times L^2_{rad}$ in $O(\delta)$. At this moment this is only an abstract norm condition not knowing the precise Sobolev space for the Cauchy values (u_0, u_1).

One imposes further that

$$a_\infty - 1 = O(\delta) \tag{6.33}$$

such that the right-had side of (6.31) now becomes $O(\delta^2)$.

One understands then the reason for introducing $g_{a_{-1}}$ in the development. It was to Taylor develop g_∞ around $a_{-1} = 1$ taking advantage of the known form for g_a as in (1.10).

For $< g_{a_{-1}}, g_\infty >$ on the right-hand side of (6.30), we express it as $< g_{-1} - g_\infty, g_\infty > + < g_\infty, g_\infty >= 1 + O(\delta)$. One then gets the existence of some uniquely determined $h = h(f_0, u_1)$ by solving the linear equation (6.30).
Moreover, again by scaling considerations, $|h| = O(\delta^2)$.

Later in the argumentation one shall be in a position to prove that h satisfies Lipschitz continuity in the variables (f_0, u_1) when precise function spaces for each of them will be specified. The apparent problem one has to manage for proving such a continuity in the variables (f_0, u_1) is that h is also a priori depending on a_∞ and on u. But this dependance is related to the one on (f_0, u_1) because changing the Cauchy values will automatically have the consequence to produce new solutions a_∞ and u.
We shall exploit this dependance to conclude in the Lipschitz continuity only in the Cauchy starting values.

One also observes, using the first equation in (6.24), that $n_-(0) = O(\delta)$.

Before going into the details of the iterative construction, we shall give some information about the domain of the operator $\sqrt{H_\infty}$ appearing in the expression for the propagator as in (6.15).

Consider $f \in \mathscr{S}(\mathbb{R}^3)$, $\mathscr{S}$ standing for the Schwartz space.

One writes using the definition of H_∞, the self-adjointness of $\sqrt{H_\infty}$ and the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ that

$$\|\sqrt{H_\infty} f\|_2^2 \lesssim \|\nabla f\|_2^2. \tag{6.34}$$

For the reverse inequality one uses integration by parts to obtain

$$\|\nabla f\|_2^2 \lesssim \|\sqrt{H_\infty} f\|_2^2 + \|V_\infty^{1/2} f\|_2^2. \tag{6.35}$$

We conclude from (6.35) and (6.34) that the operator norm of $\sqrt{H_\infty}$ is equivalent to H^1 norm on $\mathscr{S}$. Because $\sqrt{H_\infty}$ is self-adjoint, thus closed, and H^1 is the complete closure of $\mathscr{S}$ (in the H^1 norm), one concludes that $D(\sqrt{H_\infty}) = H^1$.

For future use one obtains similar relations including second derivatives

$$\begin{aligned} \|D^2 f\|_2^2 &\lesssim \|H_\infty f\|_2^2 + \|V_\infty f\|_2^2 \\ \|H_\infty f\|_2^2 &\lesssim \|D^2 f\|_2^2 + \|V_\infty f\|_2^2. \end{aligned} \tag{6.36}$$

Recursive Scheme

Implementation

We know from the development made in the section 6.2 that we are looking for a solution u decomposable into the different spectral parts as

$$u = (2k_\infty)^{-1/2}(n_+ + n_-) g_\infty + \tilde{u}, \tag{6.37}$$

satisfying the conditions (6.27).

n_+ and n_- are given respectively by (6.19) and (6.21) and the essential spectral part $\tilde{u}$ satisfies

$$\begin{aligned} \tilde{u} = cos(t\sqrt{H_\infty}) P_{g_\infty}^\perp u_0 + \frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^\perp u_1 + \\ \int_0^t \frac{sin((t-s)\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^\perp N(u, \phi_\infty)(s) ds, \end{aligned} \tag{6.38}$$

being the solution of (6.14).

One immediately observes, taking into account the form for the operator $\frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}$ as in (5.24), that it will be impossible to obtain a solution u satisfying conditions (6.27) without an additional device essentially due to the resonance term not decaying in time.

Moreover if one tries to separate the discrete spectrum part from the essential spectrum part of u for running an iterative procedure only for the essential spectrum part of the solution u, using therefore only (6.38) to build on the iterative construction, he will quickly be in trouble having to introduce scaling constants leading to higher technical difficulties which can not be consistent in the development of the procedure.

To run an iterative construction of u eliminating the resonance term at each step of the procedure, one shall have to modulate in the Aubin-Talenti parameter a.
The latter constraint shall require linearizing around some new value of the parameter a at each step of the procedure leading one to define two sequences $(u_i)_{i\in\mathbb{N}}$ and $(a_i)_{i\in\mathbb{N}}$ as follows.

Let

$$a_{-1} = 1$$

considered as the starting value for the parameter a.

At each step $i \geq 0$ of the procedure a new parameter value a_i will be defined for removing the resonance term. One therefore defines the following

$$\begin{aligned} k_i &:= k_{a_i} \\ g_i &:= g_{a_i} \\ \phi_i &:= \phi_{a_i} \\ H_i &:= H_{a_i}, \end{aligned} \tag{6.39}$$

for $i \geq -1$.

Note the important point that the expressions (6.39) are valid only in the construction of the upcoming iterative procedure. The possible issue is mainly due to the term $g_1 = g_{a_1}$ introduced in (6.39) which is not the same as g_1 in (1.10). One proceeds in this way for avoiding the complications due to the introduction of additional notations.
So the rule of thumb is that in the whole iteration procedure $g_1 = g_{a_1}$ and when the solution (a_∞, u) is determined, g_1 is the one defined in (1.10).

H_{a_i} is the operator related to the linearization around the Aubin-Talenti solution $\phi_{a_i} := \phi(., a_i)$ with parameter a_i defined in (1.9).

The first approximation for the solution u, written $u_0(t,.)$, is constructed by linearizing around $\phi(.,1) = \phi(., a_{-1})$.

$u_0(t,.)$ is then defined by

$$u_0(t,.) = cos(t\sqrt{H_{-1}})P^{\perp}_{g_{-1}}u_0 + \frac{sin(t\sqrt{H_{-1}})}{\sqrt{H_{-1}}}P^{\perp}_{g_{-1}}u_1 + \\ cos(t\sqrt{H_{-1}})((W_1 - W_{a_0})), \tag{6.40}$$

where $W_a := \phi(.,a)$, $a_0 > 0$ is to be determined for being able to remove the resonance term and c_0 is given by (5.35).
The notation W_a for the Aubin-Talenti solution is adopted here essentially to lighten notation.

One added the term $cos(t\sqrt{H_{-1}})((W_1 - W_{a_0}))$ precisely to remove the resonance part coming from the sinus evolution keeping the form of the wave solution of (6.14).
One observes that $u_0(t,.)$ is an element of the essential spectrum of H_{-1} with slightly modified Cauchy starting value $u_0(0,.)$ from u_0 to $u_0 + (W_1 - W_{a_0})$.

We shall come back in a moment to the precise equation for defining a_0.
We now continue with the iterative definition of the sequence $(u_i)_{i\geq 0}$.

Using (6.37) as a model we define $u_1(t,.)$ as

$$u_1(t,.) = (2k_0)^{-1/2}(n_{1,+} + n_{1,-})g_0 + \tilde{u}_1, \tag{6.41}$$

where the different parts of (6.41) are defined as follows.

The discrete spectrum part is given by

$$n_{1,+}(t) = -(2k_0)^{-1/2}\int_t^{\infty} e^{(t-s)k_0} < N(u_0,\phi_0), g_0 > ds \\ n_{1,-}(t) = e^{-tk_0}n_{1,-}(0) - (2k_0)^{-1/2}\int_0^t e^{-(t-s)k_0} < N(u_0,\phi_0), g_0 > ds, \tag{6.42}$$

where $n_{1,-}(0) = O(\delta)$ is arbitary.

The essential spectrum part $\tilde{u}_1$ is

$$\tilde{u}_1 = cos(t\sqrt{H_0})P^{\perp}_{g_0}u_0 + \frac{sin(t\sqrt{H_0})}{\sqrt{H_0}}P^{\perp}_{g_0}u_1 + \\ \int_0^t \frac{sin((t-s)\sqrt{H_0})}{\sqrt{H_0}}P^{\perp}_{g_0}N(u_0,\phi_0)(s)ds + cos(t\sqrt{H_0})P^{\perp}_{g_0}((W_1 - W_{a_1})). \tag{6.43}$$

One again observes the presence of the new parameter value a_1 in the last term on the right-hand side of (6.43) being defined in such a way one will be able to remove the resonance terms on the right-hand side of (6.43).

We shall come back to the precise definition of the parameter value very shortly.

$u_1(t,.)$ is thus the radiative wave part coming from the linearization around ϕ_0 with the non-linearity incorporating the known $u_0(t,.)$.

We continue the iterative procedure defining, for $i \geq 2$, u_i by

$$u_i = (2k_{i-1})^{-1/2}(n_{i,+} + n_{i,-})g_{i-1} + \tilde{u}_i, \tag{6.44}$$

where

$$\begin{aligned}
&n_{i,+}(t) = -(2k_{i-1})^{-1/2}\int_t^{\infty} e^{(t-s)k_{i-1}} < N(u_{i-1},\phi_{i-1}), g_{i-1} > ds \\
&n_{i,-}(t) = e^{-tk_{i-1}} n_{i,-}(0) - (2k_{i-1})^{-1/2}\int_0^{t} e^{-(t-s)k_{i-1}} < N(u_{i-1},\phi_{i-1}), g_{i-1} > ds \\
&\tilde{u}_i = cos(t\sqrt{H_{i-1}})P^{\perp}_{g_{i-1}} u_0 + \frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} P^{\perp}_{g_{i-1}} u_1 + \\
&\int_0^t \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} P^{\perp}_{g_{i-1}} N(u_{i-1},\phi_{i-1})(s)ds + \\
&cos(t\sqrt{H_{i-1}})P^{\perp}_{g_{i-1}}((W_1 - W_{a_i})),
\end{aligned} \tag{6.45}$$

having introduced the parameter value a_i.

Now one comes to the way of defining the sequence $(a_i)_{i\geq 0}$.

We begin with a_0.
First one writes the term $W_1 - W_{a_0} (= W_{a_{i-1}} - W_{a_0})$, using Taylor expansion, as

$$W_1 - W_{a_0} = g(1,a_0)\phi_0^{a_{-1}}(1-a_0) + O(|1-a_0| < r >^{-3}),$$

if say $|1-a_0| < \delta$.
$g(x,y)$ is strictly positive and continuous for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ being therefore uniformly controlled on an arbitrary compact neighborhood of $(1,1)$ contained in $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover $g_y(x,y) \neq 0$ on $\mathbb{R}_+ \times \mathbb{R}_+$.
$\phi_0^{a_{-1}}$ stands for the resonance associated to $H_{a_{-1}} = H_{-1}$, given for general $a > 0$ by $\phi_0^a = \partial_a \phi(.,a)$.

Taking into account the action of the projector $P^{\perp}_{g_{i-1}}$ on $\phi_0^{a_{i-1}}$ as given in (6.6) and considering (we shall come back for more precise argumentation below) that we can write

$$cos(t\sqrt{H_{-1}})\phi_0^{a_{-1}} = \phi_0^{a_{-1}}$$

and taking into account the expression for $\frac{sin(t\sqrt{H_{-1}})}{\sqrt{H_{-1}}}$ given in (5.24), one will be able to remove resonance by imposing

$$(1-a_0)g(1,a_0) + <\phi_0^{a_{-1}}, u_1> = 0.$$

This defines $a_0 > 0$ uniquely by the implicit function theorem.

One argues for general $a_i,\ i \geq 1$ as follows.
Express $\tilde{u}_i$ as

$$\begin{aligned}
\tilde{u}_i &= cos(t\sqrt{H_{i-1}})u_0 + \frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}u_1 + \\
&\int_0^\infty \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)ds - \\
&\int_t^\infty \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)ds + \\
&cos(t\sqrt{H_{i-1}})((W_1 - W_{a_i})),
\end{aligned} \tag{6.46}$$

having removed the projector $P^{\perp}_{i-1}$ for symplifying the notation, essentially considering the boundedness nature of the projector in any Sobolev norm.

Considering the resonance terms coming from $\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}u_1$ and the $\int_0^\infty$-term on the right-hand side of (6.46) one defines $a_i > 0$ by

$$(1-a_i)g(1,a_i) + <\phi_0^{a_{i-1}}, u_1> + \int_0^\infty <\phi_0^{a_{i-1}}, N(u_{i-1},\phi_{i-1})(s)> ds = 0. \tag{6.47}$$

For clarity reasons we collect the expressions for the different components of u_i after having removed the resonance as

$$\begin{aligned}
n_{+,i}(t) &= -(2k_{i-1})^{-1/2}\int_t^\infty e^{(t-s)k_{i-1}} <N(u_{i-1},\phi_{i-1}), g_{i-1}> ds \\
n_{-,i}(t) &= e^{-tk_{i-1}}n_{-,i}(0) - (2k_{i-1})^{-1/2}\int_0^t e^{-(t-s)k_{i-1}} <N(u_{i-1},\phi_{i-1}), g_{i-1}> ds \\
\tilde{u}_i &= cos(t\sqrt{H_{i-1}})u_0 + S_{i-1}(t)u_1 + \\
&\int_0^t S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds - \phi_0^{a_{i-1}}\int_t^\infty <\phi_0^{a_{i-1}}, N(u_{i-1},\phi_{i-1})(s)> ds + \\
&cos(t\sqrt{H_{i-1}})(O(|1-a_i| < r >^{-3})),
\end{aligned} \tag{6.48}$$

where we have written

$$S_{i-1}(t) := \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} - c_0(\phi_0^{a_{i-1}} \otimes \phi_0^{a_{i-1}}),$$

the linear dispersive controlled part of the sinus propagator.

Once again the projector $P_{g_{-1}}^{\perp}$ is omitted essentially because its presence will have no impact on the subsequent developments regarding its definition on corresponding resonance given in (6.6) and the fact that it is a bounded self-adjoint operator on L^2.

One observes at this point that the scalar product expression

$$< \phi_0^{a_{i-1}}, N(u_{i-1}, \phi_{i-1})(s) >$$

appearing on the fourth line in (6.48) is a priori not defined with the only condition on u_{i-1} given by (6.27) due to the form for the resonance given below in (6.51). One therefore impose an additional condition on u as

$$\| < r > u(t,.)\|_\infty = O(\delta). \tag{6.49}$$

The latter condition has to be regarded as the non-dispersive behaviour of the one dimensional free wave adapted in the present linearized setting.

The goal is now to prove that u_i satifies (6.27) for each $i \geq 0$, eventually being able to prove convergence of the sequence $(u_i)_i$, as Cauchy sequence in a well-chosen Banach spaces, to some solution u satisfying (6.4).

This shall give us, along the way, informations about the Sobolev norms for the Cauchy values (u_0, u_1).

At the same time we shall manage to prove convergence of the sequence $(a_i)_i$.

Before proceeding further we come back to the equality concerning the cosinus propagator used for eliminating the resonance

$$cos(t\sqrt{H_i})\phi_0^{a_i} = \phi_0^{a_i}. \tag{6.50}$$

This last equality can be understood in a formal sense as the consequence that $\phi_0^{a_i}$ satisfies

$$(\partial_{tt} + H_i)\phi_0^{a_i} = 0,$$

writing the formal solution of the wave equation using the propagator.

Note that the precise meaning to (6.50) is not clear because $\phi_0^{a_i} \notin L^2$. To see this, Taylor develop $\phi_0^{a_i}$ as $r \to \infty$ to obtain

$$\phi_0^{a_i} = -\frac{1}{4} 3^{1/4} a_i^{-5/4} r^{-1} + O(<r>^{-3}) \, , \; r \to \infty. \tag{6.51}$$

The only way to interpret equality (6.50) is to consider the extension of the operator $cos(t\sqrt{H_i})$ to $L^2 \cup \{\phi_0^{a_i}\}$ defined by (6.50).

One could be tempted to interpret (6.50) in the weak topology sense by trying to prove that

$$cos(t\sqrt{H_i})(\chi(./R)\phi_0^{a_i}) - \chi(./R)\phi_0^{a_i}, \tag{6.52}$$

where χ is some smooth cutoff near 0, converges to 0 in the weak topology of L^2.
This was done in another situation in [9] (p.13) but will be impossible to be proved in our case because one is not able to bound the L^2 norm of (6.52) uniformly in $R > 0$ simply because $\phi_0^{a_i} \notin L^2$.

Norms Control

The goal of this subsection is to prove the following

Proposition 6.3. *The sequence $(u_i)_{i \geq 0}$ satisfies the following uniform norm bounds with $t > 0$*

$$\begin{aligned} &\| u_{i,t} \|_\infty = O(\delta t^{-1}) \\ &\| \nabla u_{i,t} \|_{H^{1/2+2\mu}(\mathbb{R}^3)} = O(\delta) \\ &\| \nabla <r> u_{i,t} \|_{H^{1/2+2\mu}(\mathbb{R}^3)} = O(\delta) \\ &\| <r> u_{i,t} \|_\infty = O(\delta), \end{aligned} \tag{6.53}$$

where $0 < \mu \ll 1$ is small (noting that the value for μ, beside being very small compared to 1, is not totally arbitrary).

The Cauchy values (u_0, u_1) satisfy

$$\begin{aligned} &\| <r> u_0 \|_{H^{5/2(+)}(\mathbb{R}^3)} = O(\delta) \\ &\| <r> u_1 \|_{H^{3/2(+)}(\mathbb{R}^3)} = O(\delta). \end{aligned} \tag{6.54}$$

Remark 6.4. 1. The proof will force the dispersive bound (first equation in (6.53)) at stage $i \geq 1$, all the norm bounds in (6.53) being needed at stage $i-1$ to obtain the latter. One then shows that the norms control in (6.53) are valid at the stage i being therefore able to run a recursive proof, the $i = 0$ case being easily treated.

2. The regularity for the Cauchy values is the one established in the previous chapter in Lemma 5.1 and 5.3. and their control is in $O(\delta)$. The control in $O(\delta)$ is sufficient for obtaining uniformicity (in the stage i) for the bounds (6.53).

Before we embark in the proof we shall give a result and prove a technical lemma needed for the proof of Proposition 6.3.

One shall need a version of the following result, called the Gagliardo-Nirenberg inequality

Proposition 6.5. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be in $C_0^\infty(\mathbb{R}^n)$, $1 \le q, r \le \infty$ be two real numbers and $j \in \mathbb{N}$. Then if there exists $\alpha \in \mathbb{R}_+$ and $m \in \mathbb{N}_+$ such that the following holds*

$$\begin{cases} \frac{1}{p} = \frac{j}{n} + (\frac{1}{r} - \frac{m}{n})\alpha + \frac{1-\alpha}{q} \\ \frac{j}{m} \le \alpha \le 1, \end{cases} \tag{6.55}$$

then there exists a constant C not depending on u such that

$$\|D^j u\|_p \le C \|D^m u\|_r^\alpha \|u\|_q^{1-\alpha}. \tag{6.56}$$

Proof. See [4] (p.24) for a proof. □

Remark 6.6. 1. The condition (6.55) on parameters is essentially coming from the required homogeneity in the argument of the function.

2. As was pointed out before the proposition 6.5, the assumptions about the parameters appearing in its statement can be relaxed in a certain context, so that the proposition 6.5 applies even if m and j are positive real numbers.

One shall also use the next lemma in the proof of Proposition (6.3)

Lemma 6.7. *$< r >^{-3}$ satisfies the norm condition of Lemma 5.1. In other words one has $< r >^{-2} \in H^{5/2(+)}(\mathbb{R}^3)$.*

Proof of lemma 6.5: By passing in dimension $n = 1$ it suffices to show that $r < r >^{-2} \in H^{5/2(+)}(\mathbb{R}_+)$.

One remarks that the Distorted Fourier representation is depending on the linearization parameter $a > 0$ through the Fourier basis. In the present case, if one considers working at the stage i (that is the expression for $\tilde{u}_i$ in (6.48)), he will have to choose the parameter value a_{i-1}. Nevertheless, using an upcoming condition on the sequence $(a_i)_i$, the parameter $a > 0$ will in essence have almost no impact on the whole procedure.

By writing $\phi^F_{i-1}(r,\xi)$ the Fourier basis related to H_{i-1} (not to be confused with the Aubin-Talenti solution ϕ_{i-1}), one gets

$$\begin{aligned}\widehat{r<r>^{-2}}(\xi) &= \int_0^{\infty} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr = \\ \int_0^{\xi^{-1/2}} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr &+ \int_{\xi^{-1/2}}^{\infty} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr,\end{aligned} \tag{6.57}$$

remarking that it suffices to consider

$$\int_0^{\xi^{-1/2}} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr \tag{6.58}$$

when $\xi \to 0$ and

$$\int_{\xi^{-1/2}}^{\infty} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr \tag{6.59}$$

when $\xi \to \infty$.
One will then use the Taylor series representation for ϕ^F_{i-1} as described in Proposition (4.8) for (6.58) and the Jost representation given in (4.44) for the term (6.59).

As $\xi \to 0$ one then gets that

$$\int_0^{\xi^{-1/2}} r<r>^{-2}\,\phi^F_{i-1}(r,\xi)dr \approx \int_0^{\xi^{-1/2}} r<r>^{-2}\,(\phi_0^{a_{i-1}} + O(r^2\xi))dr, \tag{6.60}$$

where it is sufficient to consider the asymptotic representation of the one-dimensional resonance given by $\phi_0^{a_{i-1}} - 1 = O(<r>^{-2})$, as $r \to \infty$.

The right-hand side of (6.60) can thus be bounded as

$$\begin{aligned}&|\int_0^{\xi^{-1/2}} r<r>^{-2}\,(\phi_0^{a_{i-1}} + O(r^2\xi))dr| \lesssim \\ &\int_0^b r dr + \int_b^{\xi^{-1/2}} r^{-1}dr,\end{aligned} \tag{6.61}$$

using the bounded behaviour of $\phi_0^{a_{i-1}} + O(r^2\xi)$ and whith $b>0$ considered small compared to 1.
The apparent problem is the logarithm coming from the second integral on the right-hand side of (6.60). To overcome this difficulty we introduce inside the second integral on the right-hand side of (6.60) $\frac{r\xi^{1/2}}{r\xi^{1/2}}$ and take advantage of the regime $r\xi^{1/2} \lesssim 1$. Doing this way will produce a bound of the right-hand side of (6.60) by some constant.

One therefore concludes that (6.58) is in $L^2([0,\epsilon_2],(1+\xi^2)^{5/4(+)}\xi^{-1/2}d\xi) \approx L^2([0,\epsilon_2],\xi^{-1/2}d\xi)$.

When $\xi \to \infty$ one expresses (6.59) using the Jost representation for ϕ^F_{i-1} resulting in the approximating term

$$\int_{\xi^{-1/2}}^{\infty} < r >^{-1} \xi^{-1/2} e^{ir\xi^{1/2}} \sigma(q,r)dr. \tag{6.62}$$

To gain neagtive power in $< r >$ we shall use integration by parts taking advantage of the stability of exp as

$$e^{ir\xi^{1/2}} \approx (\xi^{-1/2})^j \frac{d^j}{dr^j}(e^{ir\xi^{1/2}}), \tag{6.63}$$

for every $j \geq 0$,
remarking that the operation will also bring more negative power in ξ.

In a first step one uses (6.63) creating enough negative power in ξ, say j_0, for $\xi^{-\frac{1}{2}(j_0+1)} \in L^2([\epsilon_2,\infty],\xi^{5/2(+)}\xi^{1/2}d\xi)$. This results in expressing (6.62) as

$$\int_{\xi^{-1/2}}^{\infty} \xi^{-\frac{1}{2}(j_0+1)} (\frac{d^{j_0}}{dr^{j_0}} e^{ir\xi^{1/2}}) < r >^{-1} \sigma(q,r)dr. \tag{6.64}$$

Then taking advantage of

$$|(\partial_r)^\alpha \sigma(q,r)| \leq < r >^{-2-\alpha} \text{ , } \alpha > 0,$$

successive integration by parts in (6.64) will permit to conclude that (6.62) is in $L^2([\epsilon_2,\infty],\xi^{5/2(+)}\xi^{1/2}d\xi)$. The boundary term at ∞ resulting from IP vanishes trivially considering the resulting negative power of $< r >$. The one at $\xi^{-1/2}$ could apparently be more problematic to manage. The only way to get rid of the boundary term at $\xi^{-1/2}$ is to consider having introduced some cutoff in the same way than was done when obtaining linear dispersive estimates in Chapter 5. In fact this cutoff is implicit in the second equality on the right-hand side of (6.57).

□

Proof of Proposition 6.2: In the first part of the proof we shall find the conditions needed to get the control $\| u_{i,t} \|_\infty = O(\delta t^{-1})$.

We remember the previously imposed norm controls given in (6.27) and (6.49) for u_{i-1}.

We start by treating the essential spectrum component of u_i given for $i \geq 1$ by

$$\begin{aligned}
&\tilde{u}_i = cos(t\sqrt{H_{i-1}})u_0 + S_{i-1}(t)u_1 + \\
&\int_0^t S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds + \phi_0^{a_{i-1}}\int_t^\infty < \phi_0^{a_{i-1}}, N(u_{i-1},\phi_{i-1})(s) > ds + \\
&cos(t\sqrt{H_{i-1}})(O(|1-a_i| < r >^{-3})),
\end{aligned} \tag{6.65}$$

noting that this is the modified expression for $\tilde{u}_{i,t}$ compared to (6.45) because we eliminated the resonance terms by defining the sequence $(a_i)_i$.

By Lemmas 5.1 and 5.3 the first two terms in (6.65), namely

$$cos(t\sqrt{H_{i-1}})u_0\ ,\ S_{i-1}(t)u_1,$$

are controlled in $\|.\|_\infty$ norm as $O(\delta t^{-1})$ if one asks for u_0 and u_1 to satisfy respectively

$$\begin{aligned}
\| < r > u_0\|_{H^{5/2}(\mathbb{R}^3)} = O(\delta) \\
\| < r > u_1\|_{H^{3/2}(\mathbb{R}^3)} = O(\delta).
\end{aligned} \tag{6.66}$$

For dealing with the term $cos(t\sqrt{H_{i-1}})(O(|1-a_i| < r >^{-3}))$ one uses the Lemma 6.7, easily concluding that

$$\|cos(t\sqrt{H_{i-1}})(O(|1-a_i| < r >^{-3}))\|_\infty = O(\delta t^{-1}),$$

by asking for

$$1 - a_i = O(\delta). \tag{6.67}$$

This condition is localizing the members of the sequence $(a_i)_{i\geq 0})$ near 1. This has the particular consequence that if a quantity is depending continuously on the parameter a the latter dependance will be uniformly controlled in the step i.

One is thus left with dealing with the two integral terms in (6.65).

We start with the one given by

$$\int_0^t S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds. \tag{6.68}$$

For bounding this term we shall assume that

$$t \to \infty,$$

which is not a limitation in the present context because we are looking for globality of solution, interested in the behaviour when $t \to \infty$.

Applying the linear dispersive estimates associated to S_{i-1} proved in Lemma 5.3 one is left to get the control of

$$\int_0^t (t-s)^{-1} \| < r > N(u_{i-1},\phi_{i-1})(s)\|_{H^{3/2(+)}(\mathbb{R}^3)} ds$$

as $O(\delta t^{-1})$.

Immediate issue is coming from the $(t-s)^{-1}$ in integrand being too much singular at $s=t$. To get around this difficulty, the solution is to break the integral in (6.68) as

$$\int_0^t = \int_0^{t-t^{-10}} + \int_{t-t^{-10}}^t, \tag{6.69}$$

controlling the first integral in (6.69) using linear dispersive estimates and the second with the properties of the sinus propagator and the Sobolev embedding $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$.

We first focus on the first integral in (6.69) bounded in $\|.\|_\infty$ norm by

$$\int_0^{t-t^{-10}} (t-s)^{-1} \| < r > N(u_{i-1},\phi_{i-1})(s)\|_{H^{3/2(+)}(\mathbb{R}^3)} ds. \tag{6.70}$$

The nonlinearity brings the following four terms to deal with

$$\begin{aligned} &< r > u_{i-1}^2 {\phi_{i-1}}^3 \\ &< r > u_{i-1}^3 {\phi_{i-1}}^2 \\ &< r > u_{i-1}^4 \phi_{i-1} \\ &< r > u_{i-1}^5. \end{aligned} \tag{6.71}$$

Remark 6.8. Before bounding those terms in $H^{3/2(+)}(\mathbb{R}^3)$-norm a word about the control of integrals of the form

$$\int_0^{t-t^{-10}} (t-s)^{-1} f(s) ds \tag{6.72}$$

in $O(t^{-1})$ is in order.
We shall prove that if $f(s) =< s >^{-1-\epsilon}$ with $\epsilon > 0$ arbitrarily small one gets the $O(t^{-1})$ control of (6.72).
To show the latter it suffices to break the integral (6.72) as

$$\int_0^{t/2} (t-s)^{-1} < s >^{-1-\epsilon} ds + \int_{t/2}^{t-t^{-10}} (t-s)^{-1} < s >^{-1-\epsilon} ds. \tag{6.73}$$

The first integral in (6.73) is controlled using the integrability of $< s >^{-1-\epsilon}$ and the second by first putting in evidence $< t >^{-1-\epsilon}$ and then controlling in $O(1)$ the logarithms coming from integration multiplied by $< t >^{-\epsilon}$, using also the fact that $t \to \infty$.

For bounding the four terms in (6.71) one could be tempted to use the paraproduct inequality ([17] (Prop. 1.1, p.105))

$$\|uv\|_{H^{s,p}} \leq \|u\|_\infty \|v\|_{H^{s,p}} + \|u\|_{H^{s,p}} \|v\|_\infty \; , \; s > 0 \; , \; 1 < p < \infty \tag{6.74}$$

valid for $u, v \in H^{s,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $n \in \mathbb{N}_+$ where $H^{s,p}$ is typical (s,p)-weighted Sobolev spaces of tempered distributions. In the present context $s = \frac{3}{2}(+)$ and $p = 2$.

The problem with this approach is that it will inevitably lead to the control of some inhomogeneous Sobolev norm of u_{i-1}. And this will clearly be impossible to control this norm recursively because, as can be seen from (6.65) and was already mentioned earlier, u_i is not in $L^2(\mathbb{R}^3)$ essentially considering the fourth term on the right-hand of (6.65) where the resonance $\phi_0^{a_{i-1}}$ appears.

Another approach has therefore to be adopted here.
The idea is to take advantage of the previously asked control (see (6.27)) for the $\|.\|_\infty$-norm of u_{i-1} in $O(\delta t^{-1})$ for bounding the derivatives part of the inhomogeneous Sobolev norm $H^{3/2(+)}(\mathbb{R}^3)$ of the terms coming from the nonlinearity, essentially using the Gagliardo-Nirenberg interpolation inequality (6.56).

Regarding the first term in (6.71) and taking into account the Remark 6.5, the following inhomogeneous Sobolev norm has to be controlled in
$O(< s >^{-1-\beta})$ with $\beta > 0$ arbitrarily small

$$\| < r > u_{i-1}(s,.)^2 {\phi_{i-1}}^3 \|_{H^{3/2(+)}}. \tag{6.75}$$

It is therefore sufficient to obtain the required control of the following two norms

$$\begin{gathered} \| < r > u_{i-1}^2 {\phi_{i-1}}^3 \|_2 \\ \| \nabla^{3/2(+)} < r > u_{i-1}^2 {\phi_{i-1}}^3 \|_2, \end{gathered} \tag{6.76}$$

with the notation $\|.\|_p = \|.\|_{L^p(\mathbb{R}^3)}$ and $1 \leq p \leq \infty$.

The first norm in (6.76) is bounded by some universal constant C_1 multiplied by

$$\|u_{i-1}\|_\infty^2, \tag{6.77}$$

C_1 being equal to $\| <r> {\phi_{i-1}}^3 \|_2$. We therefore obtain the adequate control

$$\| <r> u_{i-1}^2 {\phi_{i-1}}^3 \|_2 = O(\delta^2 <s>^{-2}).$$

The second norm in (6.76) is a little bit harder to deal with.
Application of the derivative operator $\nabla^{3/2(+)}$ leads to the control of the following two norms

$$\begin{aligned} &\| u_{i-1}^2 \nabla^{3/2(+)} (<r> {\phi_{i-1}}^3) \|_2 \\ &\| u_{i-1} (<r> {\phi_{i-1}}^3) \nabla^{3/2(+)} u_{i-1} \|_2 . \end{aligned} \tag{6.78}$$

The first norm in (6.78) is trivially controlled again using the control on the L^∞ norm of u_{i-1}.

The second term in (6.78) is, in a first step, controlled as

$$\begin{aligned} &\| u_{i-1} (<r> {\phi_{i-1}}^3) \nabla^{3/2(+)} u_{i-1} \|_2 \leq \\ &\| u_{i-1} \|_\infty \| (<r> {\phi_{i-1}}^3) \nabla^{3/2(+)} u_{i-1} \|_2 \leq \\ &C_2 \| u_{i-1} \|_\infty \| \nabla^{3/2(+)} u_{i-1} \|_{2(+)}^\gamma , \end{aligned} \tag{6.79}$$

where the last line was obtained by the application of the Hölder inequality.
γ stands for some positive number whose precise value is irrelevant in the present context.

One can now use interpolation thanks to the inequality of Gagliardo-Nirenberg for getting some dispersive control of $\|\nabla^{3/2(+)} u_{i-1}\|_{2(+)}$ using once more the dispersive bound on u_{i-1}.

A word is in order at this point to argue for the reason one can not ask for a bound of the following form

$$\|\nabla^{3/2(+)} u_{i-1}\|_2 = O(<s>^{-a}), \tag{6.80}$$

with $a > 0$, which would end the control of the the second term in (6.78) regarding the second line of (6.79).

Contrary to the wave itself, the gradient of the wave is in L^2 but its L^2-norm is not controlable in dispersive form as in (6.80). This is closely related to the conservation of energy of the wave.

To estimate what type of control one can hope for the L^2-norm of the gradient of the wave u_i (at the step i of the recursive procedure) one is considering the last term in the expression for $\tilde{u}_i$ (see (6.65)) given by

$$v(t) := cos(t\sqrt{H_{i-1}})(O(|1-a_i| <r>^{-3})).$$

We concentrate on this term because one knows that it satisfies the linearized free wave equation

$$(\partial_{tt} + H_{i-1})v = 0 \tag{6.81}$$

with Cauchy starting values given by $(O(|1-a_i| < r >^{-3}), 0)$.

Now it is not difficult to observe, using essentially the self-adjointness of $\sqrt{H_{i-1}}$ and the wave equation (6.81), that the energy of such a wave is a conserved quantity. In other words one can write

$$\|\partial_t v\|_2^2 + \|\sqrt{H_{i-1}}v\|_2^2 = \text{constant}. \tag{6.82}$$

One has now all the tools to bound $\|\nabla v(t)\|_2$.

First use (6.35) under the form

$$\|\nabla v\|_2^2 \lesssim \|\sqrt{H_{i-1}}v\|_2^2 + \|V_{i-1}^{1/2}v\|_2^2. \tag{6.83}$$

The second term on the right-hand side of (6.83) is controlled in $O(t^{-1})$ by an application of the linear dispersive estimate for the cosinus propagator and the fact that $V_{i-1}^{1/2} \in L^2$.
On the other hand, the first term on the right-hand side of (6.83) is controlled in $O(1)$ using (6.82) and the boundedness of the operator $sin(t\sqrt{H_{i-1}})$ from L^2 to itself. In more details one obtains

$$\|\partial_t v\|_2 = \|sin(t\sqrt{H_{i-1}})\sqrt{H_{i-1}}O(|1-a_i| < r >^{-3})\|_2,$$

the operators $sin(t\sqrt{H_{i-1}})$ and $\sqrt{H_{i-1}}$ commuting because the domain of $sin(t\sqrt{H_{i-1}})$ is L^2.
Finally one is taking advantage of the fact that
$< r >^{-3} \in D(\sqrt{H_{i-1}})$ by the paragraph just after (6.35).

In conlusion, using standard bounding methods, the control one is able to obtain for $\|\nabla v(t)\|_2$ is in $O(1)$.
The latter bound has therefore to be privileged.

We come back now to the last term requiring further investigation that is

$$\|\nabla^{3/2(+)}u_{i-1}\|_{2(+)}. \tag{6.84}$$

To deal with the latter term we shall use an extension, in the context of paraproducts, of the Proposition 6.5.

It is fairly easy to see that the proposition 6.5 applies with the parameters $p = 2(+)$, $r = 2$, $n = 3$, $j = \frac{3}{2} + \mu$, $m = \frac{3}{2} + 2\mu$, $q = \infty$ where $0 < \mu \ll 1$. In other words one can find $\mu > 0$ very small compared to 1 and $\alpha < 1$ but very close to 1 such that the conditions (6.55) hold.

The most important point to be noted is that $m \geq j$, that is the number of derivatives for the term on the right-hand side of (6.56) is bigger or equal than the numbers of derivatives of the term on the left-hand side of (6.56).

If one applies proposition 6.5 with the above given parameters he obtains

$$\|\nabla^{\frac{3}{2}+\mu} u_{i-1}\|_{2(+)} \leq C_2 \|\nabla^{\frac{3}{2}+2\mu} u_{i-1}\|_2^{\alpha} \|u_{i-1}\|_{\infty}^{1-\alpha}, \tag{6.85}$$

the constant $C_2 > 0$ being the one appearing in the proposition 6.5.

Summing up the discussion so far concerning the second term to be bounded in (6.78), taking into account (6.79), (6.85) and the above discussion about the L^2-norm of the gradient of the wave, one asks for the following energy norm control

$$\|\nabla u_{i-1}\|_{H^{\frac{1}{2}+2\mu}} = O(\delta). \tag{6.86}$$

The $\|.\|_{H^{3/2(+)}}$ of the other three terms coming from (6.71) are treated in the same way than the one appearing in (6.75), again considering the previous norm bounds for u_{i-1} given by (6.27), (6.49) and the last energy bound written in (6.86).
The only minor additional requirement needed for controlling the last term in (6.71) in $\|.\|_{H^{3/2(+)}}$ as $O(< s >^{-1(-)})$, that is $\| < r > u_{i-1}^5 \|_{H^{3/2(+)}}$, is the following norm bound

$$\|\nabla < r > u_{i-1}\|_{H^{\frac{1}{2}+2\mu}} = O(\delta), \tag{6.87}$$

which can in fact be reduced to the same norm control than the one in (6.86) but in dimension 1.

To finish the treatment of (6.68) it remains to bound in $O(t^{-1})$ the $\|.\|_{\infty}$ norm of the term

$$\int_{t-t^{-10}}^{t} S_{i-1}(t-s) N(u_{i-1}, \phi_{i-1})(s) ds. \tag{6.88}$$

To deal with (6.88) we shall take advantage of the Sobolev embedding $H^2(\mathbb{R}^3) \subset L^{\infty}(\mathbb{R}^3)$. Avoiding any term in integrand of the form $(t-s)^{-1}$ we shall use the following operator norm bound

$$\left\| \frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} \right\|_{2\to 2} \lesssim t\,,\ 0 < t \ll 1. \tag{6.89}$$

To prove this last fact, working in the Hilbert setting $L^2 \to L^2$, one has the following spectral representation

$$\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} = \int_{-1}^{1} \frac{sin(t\xi^{1/2})}{\xi^{1/2}} dE_{i-1}(\xi),$$

the range for ξ being restricted to the interval $[-1,1]$ containing the essential range of $\frac{sin(t\xi^{1/2})}{\xi^{1/2}}$ which is the spectrum of $\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}$. One then concludes in obtaining the bound

$$\frac{sin(t\xi^{1/2})}{\xi^{1/2}} = tO(1),$$

by geometric series expansion when $0 < t \ll 1$ and $|\xi| \leq 1$.

One writes

$$\begin{aligned}
&\int_{t-t^{-10}}^{t} S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds = \\
&\int_{t-t^{-10}}^{t} \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)ds- \\
&\int_{t-t^{-10}}^{t} \phi_0^{a_{i-1}} \otimes \phi_0^{a_{i-1}}(N(u_{i-1},\phi_{i-1})(s))ds.
\end{aligned} \tag{6.90}$$

The $\|.\|_\infty$ norm of the second integral on the right-hand side of (6.90) is easily bounded as $O(\delta t^{-1})$ essentially using (6.27) and (6.49).

The first integral on the right-hand side of (6.90) is controlled in the Sobolev norm $\|.\|_{H^2(\mathbb{R}^3)}$ as follows.

For obtaining control of the $\|.\|_{H^2(\mathbb{R}^3)}$ norm of the first integral $\int_{t-t^{-10}}^{t}(...)(s)ds$ term on the right-hand side of (6.90) in $O(\delta t^{-1})$ it is sufficient to control the following terms in $O(\delta(t-s) < s >^{-2})$ or $O(\delta < s >^{-1(-)})$

$$\begin{aligned}
&\|\frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|_{L^2}^2 \\
&\|\nabla\frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|_{L^2}^2 \\
&\|D^2\frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|_{L^2}^2.
\end{aligned} \tag{6.91}$$

The first term in (6.91) is controlled using (6.89) and the bound $N(u_{i-1},\phi_{i-1})(s) = O(\delta^2 < s >^{-2} < r >^{-3})$ using the control given in (6.27).

The second one requires the use of (6.35), the obvious boundedness of $sin((t-s)\sqrt{H_{i-1}})$ on $P_{g_{i-1}}^{\perp} L_{rad}^2$ with norm 1 and the bound for $N(s)$ used when dealing with the first term in (6.91).

The third term in (6.91) is first controlled, using the first Equation in (6.36), by

$$\begin{aligned}
&\|D^2 \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|^2_{L^2} \lesssim \\
&\|H_{i-1} \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|^2_{L^2} + \\
&\|V \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|^2_{L^2}.
\end{aligned} \tag{6.92}$$

The V term on the right-hand side of (6.92) is bounded as $O(\delta^2(t-s) < s >^{-2})$ using again (6.89), the bound for $N(s)$ used when dealing with the first term in (6.91) and the obvious fact that $V \in L^\infty(\mathbb{R}^3)$.
The H_{i-1} term on the right-hand side of (6.92) is further developed by writing

$$\begin{aligned}
&\|H_{i-1} \frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} N(u_{i-1},\phi_{i-1})(s)\|_{L^2} = \\
&\|sin((t-s)\sqrt{H_{i-1}})\sqrt{H_{i-1}} N(u_{i-1},\phi_{i-1})(s)\|_{L^2},
\end{aligned} \tag{6.93}$$

where boundedness of $sin((t-s)\sqrt{H_{i-1}})$ was used, specifically to deal with spectral integrals representing the operators $\frac{sin((t-s)\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}$ and H_{i-1} and for being able to freely permute $sin((t-s)\sqrt{H_{i-1}})$ and $\sqrt{H_{i-1}}$.

Using again the boundedness of $sin((t-s)\sqrt{H_{i-1}})$ combined with the use of (6.35) permits to conclude that the only term which requires some non-trivial control is

$$\|\nabla N(u_{i-1},\phi_{i-1})(s)\|_2.$$

The latter is controlled in $O(\delta^2 < s >^{-1(-)})$ using the interpolation method that one applied for obtaining control of the second term in (6.78). One therefore concludes using (6.27), (6.86) and (6.87).

The last term in (6.65) requiring $O(\delta t^{-1})$ control for the $\|.\|_\infty$ norm is the resonance term

$$\phi_0^{a_{i-1}} \int_t^\infty < \phi_0^{a_{i-1}}, N(u_{i-1},\phi_{i-1})(s) > ds \tag{6.94}$$

The required control is again easily obtained by applying (6.27) and (6.49).

To finish the control of the $\|.\|_\infty$-norm of the wave u_i in $O(\delta t^{-1})$ one is remaining with the discrete sepctrum part of the latter, namely $n_{i,\pm}$.
But the discrete spectrum part is also trivially controlled in $O(\delta t^{-1})$ using the decay properties of g_{i-1} and the way $n_{i,\pm}(t)$ were constructed.

Before concluding the control of $\|u_{i,t}\|_\infty$ as $O(\delta < t >^{-1})$ one has to precise the reasons for this latter control to be uniform in the stage i.

There are two reasons which could prevent the bounds in (6.53) to be uniform in the stage i.

The first issue comes from a potential multiplicative factor due to the presence of u_{i-1} in the nonlinearity term on the right-hand side of (6.65).
This multiplicative factor basically represents at each step of the procedure the number of terms to be controlled on the right-hand side of (6.65) multiplied by some universal constants. If not controlled this factor could grow with the iteration.
In fact this multiplicative factor can be controlled using that the nonlinearity term on the right-hand side of (6.65) is $O(\delta^2)$. One can thus use one of those two δ's to bound this multiplicative factor, which remains constant through iterations, by 1 at each step of the iterative procedure.

The second issue is the dependance on the stage i of the linearized operator H_i.
But thanks to the condition on the sequence a given in (6.67) one is able to uniformly control the dependance on i of the operator H_i.
The main reason for this control is that the latter dependence appears in the parameter a_i in the expressions for the Fourier basis and the latter are depending continuously on the parameter a.

The next goal is to prove that one has the same control of the remaining norms in (6.53) (except the first just controlled one) for u_i.
In other words one has to prove that

$$\begin{aligned} &\|\nabla u_i\|_{H^{1/2+2\mu}(\mathbb{R}^3)} = O(\delta) \\ &\|\nabla < r > u_i\|_{H^{1/2+2\mu}(\mathbb{R}^3)} = O(\delta) \\ &\| < r > u_i\|_\infty = O(\delta), \end{aligned} \tag{6.95}$$

We shall mainly concentrate on the essential part of the wave u_i because the discrete part satisfies almost trivially the bounds in (6.95) again taking advantage of the properties of g_{i-1} and the way $n_{i,\pm}(t)$ were constructed.

We begin with the control of the norm $\| < r > (.)\|_\infty$ of u_i.

As was already remarked, it suffices to obtain control of the L^{∞}-norm for the one-dimensional wave $r u_i$.
The latter control can then be considered as a version of the non-dispersive character of the one dimensional free wave in this non-linear setting.

It should therefore be preferable to obtain such a control if one looks toward the scattering to free wave when $t \to \infty$.

In fact the $\|r(.)\|_{\infty}$-norm of all the terms on the right-hand side of (6.65) are almost trivially controlled in $O(\delta)$ essentially using the bound for the Cauchy data given in (6.54).

For the sake of completeness we only develop the term arising from the second term on the right-hand side of (6.65) namely $rS_{i-1}(t)u_1$ as

$$
\begin{aligned}
&< u_1, \frac{\phi_0}{r} > \frac{\phi_0(r)}{r}\left(\int_0^{\epsilon_2^{1/2}t} \frac{sin(u)}{u}(\chi(\frac{u^2}{t^2})-1)du - \int_{\epsilon_2^{1/2}t}^{\infty} \frac{sin(u)}{u}du\right)+\\
&< u_1, \frac{\phi_0}{r} > \int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)\tilde{\chi}(r^2\xi)(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi+\\
&< u_1, \frac{\phi_0}{r} > \int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)(1-\tilde{\chi}(r^2\xi))(\phi(r,\xi)-\phi_0)\xi^{-1/2}d\xi + \\
&\int_0^{\epsilon_2} \frac{sin(t\xi^{1/2})}{\xi^{1/2}}\chi(\xi)(v_1(\xi)-v_1(0))\phi(r,\xi)\xi^{-1/2}d\xi+\\
&\int_{\epsilon_2}^{\infty} \frac{sin(t\xi^{1/2})}{\xi^{1/2}}(1-\chi(\xi))v_1(\xi)\phi(r,\xi)\xi^{1/2}d\xi.
\end{aligned}
\tag{6.96}
$$

One observes that the last two terms in (6.96) are treated in an even easier way they were when dealing with linear dispersive estimates in the proof of Lemma 5.3.
The next to last only needs the application of the Mean Value Theorem and then an application of the Cauchy-Schwartz inequality, the last being trivially bounded essentially using the norm controls for the Cauchy values (6.54).
For the first two terms in (6.96) one uses the uniform boundedness in ξ of the L^{∞}-norm of $\phi(r,\xi)-\phi_0$ with the possibility to control the integral of $\frac{sin(u)}{u}$ thanks to its oscillatory character as was done in the proof of Lemma 5.3.

For the second norm control in (6.95) one is again using the fact that it suffices to control the $\|\nabla(.)\|_{H^{1/2+2\mu}(\mathbb{R}^3)}$-norm of the one-dimensional wave $r u_i$.

One is therefore left with the control

$$\|\nabla u_i\|_{H^{1/2+2\mu}(\mathbb{R}^3)} = O(\delta).$$

One starts with the control of the L^2-norm of the gradient. The best option to handle this norm is to use (6.35). The term $\|V_\infty^{1/2} u_i\|_2$ is easily controlled using that $\|u_i\|_\infty = O(\delta t^{-1})$ and $V_\infty^{1/2} \in L^2$.
Then one has to control the $\|\sqrt{H_\infty}(.)\|_2$-norm of all the terms appearing on the right-hand side of (6.65).

To treat those terms one shall take advantage of the available Fourier representation for them. We start with

$$cos(t\sqrt{H_{i-1}})u_0 = \frac{1}{r}\int_0^\infty cos(t\xi^{1/2})v_0(\xi)\phi_{i-1}^F(r,\xi)\rho_{i-1}(\xi)d\xi. \tag{6.97}$$

One is then able to write

$$\sqrt{H_\infty}cos(t\sqrt{H_{i-1}})u_0 = \frac{1}{r}\int_0^\infty \xi^{1/2}cos(t\xi^{1/2})v_0(\xi)\phi_{i-1}^F(r,\xi)\rho_{i-1}(\xi)d\xi, \tag{6.98}$$

the objective being to control

$$\|\frac{1}{r}\int_0^\infty \xi^{1/2}cos(t\xi^{1/2})v_0(\xi)\phi_{i-1}^F(r,\xi)\rho_{i-1}(\xi)d\xi\|_2 \tag{6.99}$$

in $O(\delta)$.
By the Minkowski inequality for integrals one is left to consider

$$\int_0^\infty \xi^{1/2}|v_0(\xi)|\|\frac{1}{r}\phi_{i-1}^F(r,\xi)\|_2\rho_{i-1}(\xi)d\xi. \tag{6.100}$$

The L^2-norm in the integrand can be written as

$$\|\frac{1}{r}\phi_{i-1}^F(r,\xi)\|_2 \approx \left(\int_0^\infty |\phi_{i-1}^F(r,\xi)|^2 dr\right)^{1/2}, \tag{6.101}$$

using the Lebesgue measure in spherical coordinates and taking into account the radial character of the functions under consideration.

One writes (6.100) as

$$\begin{aligned}
&\int_0^\infty \xi^{1/2}|v_0(\xi)|\left(\int_0^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\rho_{i-1}(\xi)d\xi \approx \\
&\int_0^{\epsilon_2} \xi^{1/2}|v_0(\xi)|\chi(\xi)\left(\int_0^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{-1/2}d\xi + \\
&\int_{\epsilon_2}^\infty \xi^{1/2}|v_0(\xi)|(1-\chi(\xi))\left(\int_0^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{1/2}d\xi.
\end{aligned} \tag{6.102}$$

The r-integral in each of the two terms on the right-hand side of (6.102) will then be broken into two pieces one for each of the two different regimes, being thus able to use the various well-known expressions for the Fourier basis.
One obtains

$$\begin{aligned}
&\int_0^{\epsilon_2} \xi^{1/2}|v_0(\xi)|\chi(\xi)\left(\int_0^{\xi^{-1/2}} |\phi^F_{i-1}(r,\xi)|^2 dr + \int_{\xi^{-1/2}}^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{-1/2}d\xi + \\
&\int_{\epsilon_2}^\infty \xi^{1/2}|v_0(\xi)|(1-\chi(\xi))\left(\int_0^{\xi^{1/2}} |\phi^F_{i-1}(r,\xi)|^2 dr + \int_{\xi^{-1/2}}^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{1/2}d\xi.
\end{aligned} \tag{6.103}$$

By the convergence of the r-integrals due to the representations of the Fourier basis (Taylor series or Jost) and taking into account the different ranges for ξ one can restrict his attention to the following

$$\begin{aligned}
&\int_0^{\epsilon_2} \xi^{1/2}|v_0(\xi)|\chi(\xi)\left(\int_0^{\xi^{-1/2}} |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{-1/2}d\xi + \\
&\int_{\epsilon_2}^\infty \xi^{1/2}|v_0(\xi)|(1-\chi(\xi))\left(\int_{\xi^{-1/2}}^\infty |\phi^F_{i-1}(r,\xi)|^2 dr\right)^{1/2}\xi^{1/2}d\xi,
\end{aligned} \tag{6.104}$$

with the spectral densities put in evidence.

The first integral in (6.104), considering that $\phi^F_{i-1}(r,\xi) = O(1)$ (in Taylor series representation), is bounded by

$$\int_0^{\epsilon_2} \xi^{1/4}|v_0(\xi)|\xi^{-1/2}d\xi \tag{6.105}$$

being easily controlled in $O(\delta)$ using the Cauchy-Schwartz inequality and the norms control for the Cauchy data as in (6.54).

The second term in (6.104) need a convergent r-integral. To achieve this goal one has to use IP.

One writes

$$\phi^F_{i-1}(r,\xi) = a(\xi)\exp(ir\xi^{1/2})\sigma(r\xi^{1/2}, r) \tag{6.106}$$

with $a(\xi) \asymp \xi^{-1/2}$ as $\xi \to \infty$.

Inserting this expression for the Fourier basis inside the second integral in (6.104) gives the following controlling term

$$\int_{\epsilon_2}^{\infty} |v_0(\xi)| \left(\int_{\xi^{-1/2}}^{\infty} |\exp ir\xi^{1/2} \sigma(r\xi^{1/2}, r)| dr \right)^{1/2} \xi^{1/2} d\xi, \tag{6.107}$$

having considered the square of the Fourier basis (6.106) and used the fact that $\exp(ir\xi^{1/2})\sigma(r\xi^{1/2}, r) = O(1)$.

One concludes by writing the r-integral inside the term (6.107) as

$$\begin{aligned} &\int_{\xi^{-1/2}}^{\infty} |\exp ir\xi^{1/2} \sigma(r\xi^{1/2}, r)| dr = \\ &\int_{\xi^{-1/2}}^{\infty} \xi^{-1/2} |\partial_r \left(exp(ir\xi^{1/2}) \right) \sigma(r\xi^{1/2}, r)| dr. \end{aligned} \tag{6.108}$$

At this point one can use IP and the behaviour for σ given in (4.13) to bound this last integral by

$$\int_{\xi^{-1/2}}^{\infty} \xi^{-1/2} < r >^{-3} dr \tag{6.109}$$

giving finally the integral in (6.107) bounded by

$$\int_{\epsilon_2}^{\infty} \xi^{-1/4} |v_0(\xi)| \xi^{1/2} d\xi \tag{6.110}$$

again controlled in $O(\delta)$ using (6.54).

If one wants to argue in full details about the boundary terms coming from the IP just performed, he can always removed the one at ∞ taking advantage of the form for the integrand in $< r >^{-3}$ and will thus be left with a non-vanishing boundary term at $\xi^{-1/2}$.
But this is no issue because this boundary term is given by $\xi^{-1/2}|\sigma(1, \xi^{-1/2})| = O(\xi^{-1/2})$ giving in (6.110) an additional controlable integrable term using (6.54).

We now concentrate on the term $S_{i-1}(t)u_1$.
This one is a little bit more difficult to handle because one does not have a representation using the Fourier transform (see (6.96)).
The idea is then to come back to the sinus propagator writing $S_{i-1} = \frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}} - c_0 \phi_0^{i-1} \otimes \phi_0^{i-1}$ where the action of $\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}$ is given in (5.23).

Now the norm $\|\nabla(.)\|_2$ of both terms

$$\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}u_1 \\ \phi_0^{i-1}\otimes\phi_0^{i-1}(u_1) \tag{6.111}$$

are easily controlled in $O(\|u_1\|_{H^{3/2(+)}}) = O(\delta)$ using the same method than for the cos term above for the first term in (6.111) and the fact that $\nabla\phi_0^{i-1}\in L^2$ complemented with the Cauchy data norm control (6.54) for the second term in (6.111).

Using the control of the latter term one can trivially bound the integral nonlinearity term on the right-hand side of (6.65) essentially using the previously asked norm controls for u_{i-1} as in (6.27), (6.49), (6.86) and (6.87) with the convergence of the integral $\int_0^\infty < s >^{-1(-)} ds$.

Having obtained the control $\|\nabla u_i\|_2 = O(\delta)$ one is remaining with bounding the rest of the energy norm $\|\nabla u_i\|_{H^{1/2+2\mu}(\mathbb{R}^3)}$.

One can avoid further interpolation theory by asking for more regularity than the one required simply by controlling the action of the operator H_{i-1} on ∇u_{i-1}. One shall then be able to use again the Fourier representation for the propagators once commutation is handled.

If one considers the first term on the right-hand side of (6.65) the goal is to control the L^2 norm of

$$H_{i-1}\nabla\left(cos(t\sqrt{H_{i-1}})u_0\right). \tag{6.112}$$

Commuting the two operators H_{i-1} and ∇ one is left with

$$\nabla H_{i-1}\left(cos(t\sqrt{H_{i-1}})u_0\right) - \nabla(V_{i-1})\left(cos(t\sqrt{H_{i-1}})u_0\right) \tag{6.113}$$

to be controlled in $\|.\|_2$-norm as $O(\delta)$. Here V_{i-1} stand for $-5\phi(.,a_{i-1})^4$, the potential of the linearized operator at the stage $i-1$.

The second term in (6.113) is trivially handled using that $\nabla(V_{i-1})\in L^2$ and $cos(t\sqrt{H_{i-1}})u_0$ has already $\|.\|_\infty$-norm control.

One is dealing with the first term in (6.113) essentially using the same technique than the one previously applied for the control of the $\|\nabla(.)\|_2$-norm.

Using (6.35) one can write

$$\begin{aligned}&\|\nabla H_{i-1}\left(cos(t\sqrt{H_{i-1}})u_0\right)\|_2^2 \lesssim\\ &\|\sqrt{H_{i-1}}H_{i-1}\left(cos(t\sqrt{H_{i-1}})u_0\right)\|_2^2 +\\ &\|V_{i-1}^{1/2}H_{i-1}\left(cos(t\sqrt{H_{i-1}})u_0\right)\|_2^2,\end{aligned} \tag{6.114}$$

the two terms on the right-hand side of (6.115) being handled using the Fourier transfrom representation of $cos(t\sqrt{H_{i-1}})u_0$ with the same bounding technique than the one used before. We omit the easy calculations.

One is therefore left with the term of the form $S_{i-1}(t)u_1$. The method is word by word the same as for the case of the cos term just treated with the minor difference that one has again to consider to come back to the expression involving the sin propagator.
The only point to be noted is that one is allowed to write

$$\begin{aligned}&\|\nabla H_{i-1}\left(S_{i-1}(t)\right)\|_2^2 \lesssim\\ &\|\sqrt{H_{i-1}}H_{i-1}\left(\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}u_1\right)\|_2^2 +\\ &\|V_{i-1}^{1/2}H_{i-1}\left(\frac{sin(t\sqrt{H_{i-1}})}{\sqrt{H_{i-1}}}u_1\right)\|_2^2\end{aligned} \tag{6.115}$$

because $H_{i-1}\phi_0^{i-1} = 0$.

Putting all the pieces together, one can therefore run the recursive norms control of the sequence $(u_i)_{i\geq 0}$ beginning with the modified form (after removing the resonance terms) for $u_{0,t}$ given by

$$\begin{aligned}&u_0(.,t) = cos(t\sqrt{H_{-1}})u_0 + S_{-1}(t)u_1 +\\ &cos(t\sqrt{H_{-1}})(|1-a_0| < r >^{-3}),\end{aligned} \tag{6.116}$$

which clearly satisfies the norm bounds (6.53) by the same techniques used so far.

This ends the proof of proposition 6.3. □

Convergence

The goal of this subsection is to obtain convergence results concerning the sequences u and a.

The a priori difficulty lies in the fact that these two sequences influence each other.
We shall therefore show that the couple $(a,u) = ((a_i,u_i))_{i\geq 0}$ is a Cauchy sequence in suitable

Banach norm.

Let $0 < \epsilon < 1$ be given.
We shall keep the value for ϵ fixed for the rest of the proof of Theorem 6.1.

One gets the following

Proposition 6.9. *The sequence (a, u) is Cauchy in the Banach norm given by*

$$b + \|f\|_{\mathscr{Y}} + \|f\|_{\mathscr{Z}}, \tag{6.117}$$

where $(b, f) \in (1-\delta, 1+\delta) \times \mathscr{Y} \cap \mathscr{Z}$.
$\mathscr{Y}$ and $\mathscr{Z}$ are given by
$\mathscr{Y} := \{g \in C([0,\infty), L^{\infty}(\mathbb{R}^3)) \,:\, \|g(t,.)\|_{\infty} = O(< t >^{-1})\,,\ t \geq 0\}$ *with the norm* $\|g\|_{\mathscr{Y}} := sup_{s\geq 0} < s >^{\epsilon} \|g(s,.)\|_{\infty}$ *and*
$\mathscr{Z} := \{g \in C([0,\infty), \dot{H}^1(\mathbb{R}^3)) \,:\, \|\nabla g(t,.)\|_{H^{1/2+2\mu}} = O(1)\,,\ t \geq 0\}$ *with the norm* $\|g\|_{\mathscr{Z}} := sup_{s\geq 0} \|\nabla g(s,.)\|_{H^{1/2+2\mu}}$

Proof. We first recall the definition of $(a_i)_i$ as

$$\begin{cases} a_{-1} = 1 \\ a_0 = 1 - < \phi_0^{a_{-1}}, u_1 > \\ a_i - 1 = < \phi_0^{a_{i-1}}, u_1 > + \int_0^{\infty} < \phi_0^{a_{i-1}}, N(u_{i-1}, \phi_{i-1})(s) > ds\,,\ i \geq 1. \end{cases} \tag{6.118}$$

One observes that we omitted the presence of the function g present in (6.47) mainly because $g(1, a_i)$ is uniformly controlled in i by its very properties described at the bottom of the page 89 and by the condition on the sequence a given in (6.67).

One writes $a_i - a_{i-1}$, $i \geq 2$, as

$$\begin{aligned} &\left(< \phi_0^{a_{i-1}}, u_1 > - < \phi_0^{a_{i-2}}, u_1 >\right) + \\ &\left(\int_0^{\infty} < \phi_0^{a_{i-1}}, N(u_{i-1}, \phi_{i-1})(s) > ds - \int_0^{\infty} < \phi_0^{a_{i-2}}, N(u_{i-2}, \phi_{i-2})(s) > ds\right). \end{aligned} \tag{6.119}$$

The first paranthesis of (6.119) can be expressed as

$$< \phi_0^{a_{i-1}}, u_1 > - < \phi_0^{a_{i-2}}, u_1 > = < \phi_0^{a_{i-1}} - \phi_0^{a_{i-2}}, u_1 > . \tag{6.120}$$

Taylor expansion implies that $\phi_0^{a_{i-1}} - \phi_0^{a_{i-2}} = O(|a_{i-1} - a_{i-2}| < r >^{-3})$ whence (6.120) is

$$O(\delta |a_{i-1} - a_{i-2}|)$$

by the Cauchy-Schwartz inequality using essentially that $< r >^{-3} \in L^2(\mathbb{R}^3)$ and the Cauchy values norm bounds given in (6.54).

Now one concentrates on the second paranthesis on the right-hand side of (6.119) expressing it as

$$\begin{aligned}
&\int_0^\infty <\phi_0^{a_{i-1}}, N(u_{i-1},\phi_{i-1})(s)> ds - \int_0^\infty <\phi_0^{a_{i-2}}, N(u_{i-2},\phi_{i-2})(s)> ds = \\
&\int_0^\infty <\phi_0^{a_{i-1}} - \phi_0^{a_{i-2}}, N(u_{i-1},\phi_{i-1})(s)> ds + \\
&\int_0^\infty <\phi_0^{a_{i-2}}, N(u_{i-1},\phi_{i-1})(s) - N(u_{i-2},\phi_{i-2})(s)> ds.
\end{aligned} \tag{6.121}$$

The first term on the right-hand side of (6.121) is

$$O(\delta^2|a_{i-1}-a_{i-2}|) \subset O(\delta|a_{i-1}-a_{i-2}|)$$

using the norm controls (6.53).

For the second term on the right-hand side of (6.121) we shall use the well-known decomposition in factors such that for every $u, v \in \mathbb{C}$ and every $n \geq 1$ one gets

$$u^n - v^n \approx (u-v)^n + (u-v)^{n-1} v + ... + (u-v) v^{n-1}, \tag{6.122}$$

approximation equality meaning modulo some (irrelevant) constants.

Applying (6.122) to $N(u_{i-1},\phi_{i-1})(s) - N(u_{i-2},\phi_{i-2})(s)$, only treating in details the first term from the nonlinearity N, expressing it as

$$u_{i-1}^2\phi_{i-1}^3 - u_{i-2}^2\phi_{i-2}^3 = u_{i-1}^2(\phi_{i-1}^3 - \phi_{i-2}^3) + (u_{i-1}^2 - u_{i-2}^2)\phi_{i-2}^3,$$

one observes that the second term on the right-hand side of equality in (6.121) is controlled as

$$\begin{aligned}
&|\int_0^\infty <\phi_0^{a_{i-2}}, N(u_{i-1},\phi_{i-1})(s) - N(u_{i-2},\phi_{i-2})(s)> ds| \leq \\
&O(\delta\left(|a_{i-1}-a_{i-2}| + sup_{s\geq 0}\left(<s>^\epsilon \|(u_{i-1}-u_{i-2})(s,.)\|_\infty\right)\right)).
\end{aligned} \tag{6.123}$$

We note that $sup_{s\geq 0}(<s>^\epsilon \|(u_{i-1}-u_{i-2})(s,.)\|_\infty)$ is well-defined by (6.53) the factor term $<s>^\epsilon$ used to make the integral $\int_0^\infty(...)ds$ convergent.

One thus concludes that

$$|a_i - a_{i-1}| \leq O(\delta(|a_{i-1}-a_{i-2}| + sup_{s\geq 0}\left(<s>^\epsilon \|u_{i-1,s} - u_{i-2,s}\|_\infty\right))). \tag{6.124}$$

We now turn to the control of $u_i - u_{i-1}$ naturally beginning in the Banach norm given by

$$\|f\|_{\mathscr{Y}} := sup_{s\geq 0}\left(< s >^{\epsilon} \|f(s,.)\|_{\infty}\right), \tag{6.125}$$

where $\mathscr{Y} := \{g \in C([0,\infty), L^{\infty}(\mathbb{R}^3)) \,:\, \|g(t,.)\|_{\infty} = O(< t >^{-1})\,,\ t > 0\}$.

One writes $u_i - u_{i-1}$ as

$$\begin{aligned}
&(2k_{i-1}^{-1/2} - 2k_{i-2}^{-1/2})(n_{i,+} + n_{i,-})g_{i-1} + \\
&2k_{i-2}^{-1/2}((n_{i,+} - n_{i-1,+}) + (n_{i,-} - n_{i-1,-}))g_{i-1} + \\
&2k_{i-2}^{-1/2}(n_{i-1,+} + n_{i-1,-})(g_{i-1} - g_{i-2}) + \\
&\tilde{u}_i - \tilde{u}_{i-1}
\end{aligned} \tag{6.126}$$

where for the discrete spectrum part, in accordance with (6.45), one has

$$\begin{aligned}
&n_{i,+}(t) - n_{i-1,+}(t) = \\
&-(2k_{i-1})^{-1/2}\int_t^{\infty} e^{(t-s)k_{i-1}} < N(u_{i-1},\phi_{i-1}), g_{i-1} > ds + \\
&(2k_{i-2})^{-1/2}\int_t^{\infty} e^{(t-s)k_{i-2}} < N(u_{i-2},\phi_{i-2}), g_{i-2} > ds,
\end{aligned} \tag{6.127}$$

and

$$\begin{aligned}
&n_{i,-}(t) - n_{i-1,-}(t) = \\
&\left(e^{-tk_{i-1}} n_{i,-}(0) - e^{-tk_{i-2}} n_{i-1,-}(0)\right) + \\
&(2k_{i-2})^{-1/2}\int_0^{t} e^{-(t-s)k_{i-2}} < N(u_{i-2},\phi_{i-2}), g_{i-2} > ds - \\
&(2k_{i-1})^{-1/2}\int_0^{t} e^{-(t-s)k_{i-1}} < N(u_{i-1},\phi_{i-1}), g_{i-1} > ds
\end{aligned} \tag{6.128}$$

For the essential spectrum part one gets using the expression (6.65)

$$\begin{aligned}
&\tilde{u}_i - \tilde{u}_{i-1} = \\
&cos(t\sqrt{H_{i-1}})u_0 - cos(t\sqrt{H_{i-2}})u_0 + \\
&S_{i-1}(t)u_1 - S_{i-2}(t)u_1 + \\
&\int_0^t S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds - \\
&\int_0^t S_{i-2}(t-s)N(u_{i-2},\phi_{i-2})(s)ds + \\
&cos(t\sqrt{H_{i-1}})(O(|1-a_i| < r >^{-3})) - \\
&cos(t\sqrt{H_{i-2}})(O(|1-a_{i-1}| < r >^{-3})).
\end{aligned} \tag{6.129}$$

Regarding the discrete spectrum part we shall treat in details the case of n_-. n_+ follows using the same type of considerations.

For dealing with the first paranthesis term on the right-hand side of (6.128) namely

$$e^{-tk_{i-1}} n_{i,-}(0) - e^{-tk_{i-2}} n_{i-1,-}(0), \tag{6.130}$$

one Taylor expands $e^{-tk_{i-2}}$ around $k = k_{i-1}$ obtaining (modulo constants)

$$\begin{aligned} &e^{-tk_{i-1}} n_{i,-}(0) - e^{-tk_{i-2}} n_{i-1,-}(0) = \\ &e^{-tk_{i-1}}(n_{i,-}(0) - n_{i-1,-}(0)) + O((k_{i-1} - k_{i-2}) n_{i-1,-}(0)) \;,\; t \to \infty \end{aligned} \tag{6.131}$$

the O notation being justified due the representation (1.10), the hypothesis about $(a_i)_i$ as in (6.67) and the time decay of the successive derivatives as $t^n e^{-bt} \to 0$ when $t \to \infty$ for $n \in \mathbb{N}, n \geq 1$ and $b > 0$.

Now we can write the first term on the right-hand side of (6.131) as

$$O(\delta|a_{i-1} - a_{i-2}|)$$

by taking advantage of the fact that in our recursive construction $n_{i,-}(0)$ is free up to satisfy $n_{i,-}(0) = O(\delta)$. Applying the condition (6.67), one thus defines it as

$$n_{i,-}(0) = \delta(1 - a_{i-1}),$$

which is in fact $O(\delta^2) \subset O(\delta)$.

For the second term on the right-hand side of (6.131) one Taylor expands $(k_{i-1} - k_{i-2})$ using the scaling properties (1.10) and by an additional use of condition (6.67) one obtains that

$$(k_{i-1} - k_{i-2}) t e^{-tk_{i-1}} n_{i-2,-}(0) = O(\delta|a_{i-1} - a_{i-2}|), \tag{6.132}$$

the δ coming again from the fact that $n_{i-2,-}(0) = O(\delta)$.

The same considerations as those used when dealing with the sequence a can be applied for controlling the absolute value of the integral terms in (6.128).

It is therefore clear, taking into account the time decay coming from the integrals in (6.128) of the form $\int_t^\infty (...)(s)ds$, that the member of (6.126) containing $n_{i,-} - n_{i-1,-}$ can be controlled in the $\|.\|_{\mathscr{Y}}$ as

$$\|2k_{i-2}^{-1/2}(n_{i,-} - n_{i-1,-})(t)g_{i-1}\|_{\mathscr{Y}} = O(\delta(|a_{i-1} - a_{i-2}| + \|u_{i-1} - u_{i-2}\|_{\mathscr{Y}})). \tag{6.133}$$

We come now to the essential sepctrum part of $\tilde{u}_i - \tilde{u}_{i-1}$.

We shall first treat the first term in (6.129), namely

$$cos(t\sqrt{H_{i-1}})u_0 - cos(t\sqrt{H_{i-2}})u_0.$$

The difficulty lies in the treatment of the operator

$$cos(t\sqrt{H_{i-1}}) - cos(t\sqrt{H_{i-2}}), \tag{6.134}$$

which require the analysis of the difference of the Fourier bases ϕ^F_{i-1} and ϕ^F_{i-2}.

Considering that the spectral density is independant of the stage i in the asymptotic limit given in Proposition 4.48 one then expresses the action of the operator (6.134) on u_0 as

$$\begin{aligned}&(cos(t\sqrt{H_{i-1}}) - cos(t\sqrt{H_{i-2}}))u_0 = \\ &\frac{1}{r}\int_0^{\epsilon_2} cos(t\xi^{1/2})v_0(\xi)(\phi^F_{i-1}(r,\xi) - \phi^F_{i-2}(r,\xi))\chi(\xi)\xi^{-1/2}d\xi + \\ &\frac{1}{r}\int_{\epsilon_2}^{\infty} cos(t\xi^{1/2})v_0(\xi)(\phi^F_{i-1}(r,\xi) - \phi^F_{i-2}(r,\xi))(1-\chi(\xi))\xi^{1/2}d\xi,\end{aligned} \tag{6.135}$$

the remaining work being to find how to deal with the difference $\phi^F_{i-1}(r,\xi) - \phi^F_{i-2}(r,\xi)$ such that one can bound the $\|.\|_{\mathscr{Y}}$ of (6.135) in an appropriate way.
In other words one has to express the difference of Fourier bases such that he will be able to reproduce the linear dispersive behaviour proved in Lemma 5.1.

Breaking the two integrals on the right-hand side of (6.135) using a cutoff expression of the form $\tilde{\chi}(r^2\xi)$ as was done in the proof of lemma 5.1 one is able to write the difference of Fourier bases in the following form

$$\begin{aligned}&r\phi_0^{a_{i-1}} + O_{i-1}(r^2\xi) - r\phi_0^{a_{i-2}} + O_{i-2}(r^2\xi)\ ,\ r^2\xi \le 1 \\ &a(\xi)(f_{i-1,+}(r,\xi) - f_{i-2,+}(r,\xi)) + \overline{a(\xi)}(f_{i-1,-}(r,\xi) - f_{i-2,-}(r,\xi))\ ,\ r^2\xi > 1\end{aligned}, \tag{6.136}$$

with $f_{i-1,+}$ being the Jost solution at stage $i-1$ and where we neglect the dependance of the coeficients $a(\xi)$ on the stage i because we shall exclusively use their asymptotic expressions given in (4.49).

As a reminder we give the form of the Jost solution at the stage $i-1$

$$e^{ir\xi^{1/2}}\sigma_{i-1}(r\xi^{1/2},r), \tag{6.137}$$

where clearly the i inside the exponential function is the complex number and σ satifies the behaviour given in (4.12) and (4.13).

Observing that the O_i notation (at the stage i) on the first line of (6.136) was obtained using exclusively the resonance $r\phi_0^{a_i}$, one shall concentrate on the dependance of the resonance (in dimension 1) $r\phi_0^a$ and the asymptotic sum σ_i on the parameter $a>0$.

We already remarked that the expression in the first line of (6.136) is only needed in the range $r\to 0$ when $r^2\xi\leq 1$. In this range for r the resonance $r\phi_0^a$ has the form $O(a^{-3/4}r)$, having Taylor expanded around $r=0$ the Aubin-Talenti (multiplied by r) solution given in the form (1.3) and then derived in a.
On the other hand the second line in (6.136) is only considered in the $r\to\infty$ range.
Taylor expanding in the range $r\to\infty$ the functions $\psi^+_{i,j}(r)$, $j\in\mathbb{N}$, appearing in the expression for σ_i in (4.12) and taking into account the hypothesis about the sequence $(a_i)_i$ as in (6.67), one concludes that (6.136) can be approximated by

$$\begin{aligned} &O(r|a_{i-1}-a_{i-2}|)+O(|a_{i-1}-a_{i-2}|r^2\xi)\ ,\ r^2\xi\leq 1\ , r\to 0\\ &a(\xi)(a_{i-1}-a_{i-2})e^{ir\xi^{1/2}}\tilde{\sigma}(r\xi^{1/2},r)\ ,\ r^2\xi>1\ ,\ r\to\infty, \end{aligned} \tag{6.138}$$

where $\tilde{\sigma}$ satisfies

$$\tilde{\sigma}(q,r)\sim\sum_{j=1}^{\infty}q^{-j}\tilde{\psi}^+_j(r) \tag{6.139}$$

in the following sense

$$\begin{cases} sup_{r>0}<r>^4|(r\partial_r)^\alpha(q\partial_q)^\beta[\tilde{\sigma}(q,r)-\sum_{j=1}^{j_0}q^{-j}\tilde{\psi}^+_j(r)]|\leq c_{\alpha,\beta,j_0}q^{-j_0-1}\\ \text{for all } \alpha,\beta,j_0\geq 1, \end{cases} \tag{6.140}$$

where the $\tilde{\psi}^+_j(r)$ are symbol of order -4 satisfying

$$sup_{r>0}<r>^4|(r\partial_r)^\alpha\tilde{\psi}^+_j(r)|<\infty$$

for all $\alpha\geq 0$ and $j\geq 1$.

It is therefore evident considering the form for the Fourier bases difference given in (6.138) and proceeding in exactly the same way than was done in the proof of Lemma 5.1 that

$$\|(cos(t\sqrt{H_{i-1}})-cos(t\sqrt{H_{i-2}}))u_0\|_{\mathcal{Y}}=O(\delta|a_{i-1}-a_{i-2}|), \tag{6.141}$$

δ on the right-hand side of (6.141) resulting from the Cauchy values norm controls given in (6.54).

One obtains similarly that

$$\|S_{i-1}(t)u_1 - S_{i-2}(t)u_1\|_{\mathscr{Y}} = O(\delta|a_{i-1} - a_{i-2}|). \tag{6.142}$$

The control of the nonlinear part given by

$$\| \int_0^t S_{i-1}(t-s)N(u_{i-1},\phi_{i-1})(s)ds - \int_0^t S_{i-2}(t-s)N(u_{i-2},\phi_{i-2})(s)ds\|_{\mathscr{Y}} \tag{6.143}$$

will lead to find the total Banach norm needed to obtain the Cauchy property for the sequence (a,u).

Not giving all the tedious calculations and verification details, one obtains that the term in (6.143) is controlled in

$$O(\delta(|a_{i-1} - a_{i-2}| + \|u_{i-1} - u_{i-2}\|_{\mathscr{Y}} + sup_{s\geq 0}\|\nabla u_{i-1}(s,.) - \nabla u_{i-2}(s,.)\|_{H^{1/2+2\mu}}) \tag{6.144}$$

the argument of the O-notation being well-defined due the the norm controls proved in Proposition 6.3.

One has then also to control the difference $u_i - u_{i-1}$ in the energy norm $\|.\|_{\mathscr{Z}}$ given by

$$\|f\|_{\mathscr{Z}} := sup_{s\geq 0}(\|\nabla f_s\|_{H^{1/2+2\mu}}), \tag{6.145}$$

where $\mathscr{Z} := \{g \in C([0,\infty), \dot{H}^1(\mathbb{R}^3)) \,:\, \|\nabla g_t\|_{H^{1/2+2\mu}} = O(1)\,,\ t \geq 0\}$.
This is done essentially applying the same techniques and results used so far in the proof supplemented by some calculations similar to those performed in pages 105 to 108.

Moreover one argues that all the $O(.)$ controls done so far are uniform with respect to the stage i.
The first reason is the fact that one can use the nonlinearity scale in $O(\delta^2)$ giving thus the possibility to use one the two δ's to control, at each step i of the procedure, a potential constant multiplicative factor.
A second argument is coming from the condition on the sequence a given in (6.67).

One concludes the proof by observing that $|a_1 - a_0| = \|u_1 - u_0\|_{\mathscr{Y}} = \|u_1 - u_0\|_{\mathscr{Z}} = O(\delta^2)$ using essentially the norm controls (6.53).

This ends the proof of Proposition 6.9.

□

Remark 6.10. 1. The fact that we were able to express the difference of Fourier bases as in (6.138) in a strongly similar form than the Fourier basis itself (for a fixed value of the parameter $a>0$) is crucial for esimating the $\|.\|_{\mathcal{Y}}$ norm of the left-hand side of (6.135) in a similar way than was performed when obtaining linear dispersive estimates in the preceding chapter.

2. One remarks that $\tilde{\sigma}$ has also the asymptotic sum representation but with coefficient functions $\tilde{\psi}_j^+$, with $j \geq 1$, being symbols of order -4 in place of the -2 order satisfied by the coefficient functions in the Jost expression of the Fourier basis itself.

Applying the proposition 6.9 one obtains a couple $(a_\infty, u) \in (1-\delta, 1+\delta) \times \mathcal{Y} \cap \mathcal{Z}$ which is the limit of the sequence $(a_i, u_i)_i$ in the indicated Banach norm.

Remark 6.11. There is a possible issue coming from the fact that the notation u is used to describe the sequence $(u_i)_i$ and its limit. As a (logical) rule of thumb if the notation u appears until the end of the proof of Proposition (6.9) this is to denote the sequence $(u_i)_i$ and from now on u will denote its limit.

Using exactly the same techniques than in the proof of Proposition 6.9 we prove that u is expressed as

$$u = (2k_\infty)^{-1/2}(n_+ + n_-)g_\infty + \tilde{u}, \tag{6.146}$$

where k_∞ and g_∞ are the already defined quantities associated to the parameter value a_∞. Moreover n_+, n_- and $\tilde{u}$ are respectively given by

$$\begin{aligned}
&n_+(t) = -(2k_\infty)^{-1/2}\int_t^\infty e^{(t-s)k_{i-1}} < N(u,\phi_\infty), g_\infty > ds \\
&n_-(t) = e^{-tk_\infty} n_-(0) - (2k_\infty)^{-1/2}\int_0^t e^{-(t-s)k_\infty} < N(u,\phi_\infty), g_\infty > ds \\
&\tilde{u} = cos(t\sqrt{H_\infty})u_0 + S_\infty(t)u_1 + \\
&\int_0^t S_\infty(t-s)N(u,\phi_\infty)(s)ds + \phi_0^{a_\infty}\int_t^\infty < \phi_0^{a_\infty}, N(u,\phi_\infty)(s) > ds + \\
&cos(t\sqrt{H_\infty})(O(|1-a_\infty| < r >^{-3})).
\end{aligned} \tag{6.147}$$

Besides a_∞ satisfies

$$a_\infty - 1 = < \phi_0^{a_\infty}, u_1 > + \int_0^\infty < \phi_0^{a_\infty}, N(u,\phi_\infty)(s) > ds. \tag{6.148}$$

The equations (6.148) and (6.147) satisfied by (a_∞, u) will be used when proving the Lipschitz continuity of h in the next section. One can in fact consider this system of equations as defining a fixed point for some contracting function Φ in a Banach space setting.

One has therefore proved the existence of a global in positive time solution of (1.1) of the form $\phi_\infty + u(r,t)$.

Coming back to the starting equation for the essential spectrum part of the solution one can express $\tilde{u}$ as

$$\begin{aligned}\tilde{u} = & cos(t\sqrt{H_\infty})u_0 + \frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}u_1 + \\ & \int_0^t \frac{sin((t-s)\sqrt{H_\infty})}{\sqrt{H_\infty}}N(u,\phi_\infty)(s)ds + \\ & cos(t\sqrt{H_\infty})((W_1 - W_{a_\infty})).\end{aligned} \tag{6.149}$$

Remark 6.12. (6.149) brings the additional information that the Cauchy values are of the form

$$(u_0 + (W_1 - W_{a_\infty}), u_1), \tag{6.150}$$

concluding that a correction to the Cauchy starting value u_0 has to be made for obtaining the global solution u.

We shall come back shortly for a complete characterization of the set of admissible Cauchy values eventually proving the existence of a codimension one stable manifold associated to the linearized wave equation (1.5).

Stable manifold

In this section we shall collect all the relevant informations about the Cauchy starting values $(u_0, u_1) \in L^2_{rad} \times L^2_{rad}$ needed to solve the linearized wave equation (1.5).

Taking into account the previous conditions established in Section 6.2 and those collected in Proposition 6.3 one is thus able to conclude that the set of admissible Cauchy values forms a manifold, called the *stable manifold* relative to the linearized equation (1.5) with starting values (u_0, u_1).
This manifold is parametrized by the pair of functions (f_0, u_1), where f_0 was introduced in (6.29), such that they satisfy the following *codimension one* condition

$$< g_{-1}, k_{a_{-1}} f_0 + u_1 >= 0. \tag{6.151}$$

Moreover, by Proposition 6.3, (u_0, u_1) has to satisfy $\| < r > u_0 \|_{H^{5/2(+)}(\mathbb{R}^3)} = O(\delta)$ and $\| < r >$

$u_1\|_{H^{3/2(+)}(\mathbb{R}^3)} = O(\delta)$ which translates into the same conditions on the pair (f_0, u_1) as

$$\begin{aligned} &\| < r > f_0\|_{H^{5/2(+)}(\mathbb{R}^3)} = O(\delta) \\ &\| < r > u_1\|_{H^{3/2(+)}(\mathbb{R}^3)} = O(\delta), \end{aligned} \tag{6.152}$$

due to the fact that in the expression for u_0 given by (6.40) $h(f_0, u_1) = O(\delta^2)$.

Using the set of pairs (f_0, u_1) satisfying (6.151), (6.152) as a coordinate set, the stable manifold is written under the graph form as

$$(f_0 + h(f_0, u_1)g_1 + (W_1 - W_{a_\infty(f_0,u_1)}), u_1) \tag{6.153}$$

where $a_\infty(f_0, u_1)$, the limit parameter obtained as a consequence of Proposition 6.9, is a priori depending on (f_0, u_1).
One notes the correction for the Cauchy value u_0 pointed out in Remark 6.12.

One reminds that $h(f_0, u_1)$ is $O(\delta^2)$ and is given by the solution of the first-order linear equation (6.30).

We shall finish this section by investigating a little further the continuity properties of the function $h = h(f_0, u_1)$.

We start by considering h defined on $\mathscr{U}$ as in the following

$$\begin{aligned} &\mathscr{U} := \mathscr{B}_{1,\delta} \times \mathscr{B}_{2,\delta} \times (1-\delta, 1+\delta) \times \mathscr{T} \mapsto \mathbb{C} \\ &h : (f, g, a, u) \to h(f, g, a, u), \end{aligned} \tag{6.154}$$

where $\mathscr{T} := \mathscr{Y} \cap \mathscr{Z}$, $\mathscr{B}_{1,\delta} := \{f \in L^2(\mathbb{R}^3) : \| < r > f\|_{H^{5/2(+)}(\mathbb{R}^3)} < \delta\}$ and $\mathscr{B}_{2,\delta} := \{g \in L^2 : \| < r > g\|_{H^{3/2(+)}(\mathbb{R}^3)} < \delta\}$.
$\mathscr{Y}$ and $\mathscr{Z}$ are defined in the statement of Proposition 6.9.

Ultimately the domain of h will be represented by a quadruplet (f_0, u_1, a_∞, u) such that (f_0, u_1) are the starting Cauchy values, a_∞ is the limit parameter and u is the solution of the linearized problem with Cauchy values (f_0, u_1).

Introducing the metric $d_{\mathscr{U}}$ (on $\mathscr{U}$) induced by the norm

$$\begin{aligned} &\|(f, g, a, u)\|_{\mathscr{U}} := \\ &\| < r > f\|_{H^{5/2(+)}(\mathbb{R}^3)} + \| < r > g\|_{H^{3/2(+)(\mathbb{R}^3)}} + a + \|u\|_{\mathscr{Y}} + \|u\|_{\mathscr{Z}}, \end{aligned} \tag{6.155}$$

h is shown to be Lipschitz continuous using $d_{\mathscr{U}}$.
To see the latter it is enough to express $h(f,g,a,u) - h(\tilde{f},\tilde{g},\tilde{a},\tilde{u})$
as

$$\big(h(f,g,a,u) - h(\tilde{f},\tilde{g},a,u)\big) + \big(h(\tilde{f},\tilde{g},a,u) - h(\tilde{f},\tilde{g},\tilde{a},\tilde{u})\big) \tag{6.156}$$

and use standard bounding procedure applied to the solution of (6.30).

Now one introduces relations between the different components of $\mathscr{U}$.

The link between the variables of h lies in the fact that a which will eventually be a_∞ and u (considered from now as the solution of the linearized problem associated to the pair (f_0, u_1)) are dependant on (f_0, u_1).
It should therefore be possible to obtain Lipschitz continuity of h in only the two variables (f, g) which will eventually be considered to be the starting Cauchy values (f_0, u_1).

The main idea is to regard the whole problem of finding the solution for the linearized problem showing free dispersive time decay, that is the pair (a_∞, u), as a fixed point procedure.

One therefore starts by defining the map Φ whose fixed point property will have to be investigated.

One writes

$$\Phi : \begin{cases} \mathscr{U} \to (1-\delta, 1+\delta) \times \mathscr{T} \\ (f_0, u_1, b, v) \mapsto (a, w), \end{cases} \tag{6.157}$$

with (a, u) satisfying the following two equations

$$a - 1 = < \phi_0^b, u_1 > + \int_0^\infty < \phi_0^b, N(v, \phi_b)(s) > ds, \tag{6.158}$$

and

$$\begin{aligned} & w = (2k_b)^{-1/2}(n_{+,w} + n_{-,w})g_b + \tilde{w}, \text{ where} \\ & n_{+,w}(t) = -(2k_b)^{-1/2} \int_t^\infty e^{(t-s)k_b} < N(v,\phi_b), g_b > ds \\ & n_{-,w}(t) = e^{-tk_b} n_{-,w}(0) - (2k_b)^{-1/2} \int_0^t e^{-(t-s)k_b} < N(v,\phi_b), g_b > ds \\ & \tilde{w}(t,.) = cos(t\sqrt{H_b}) f_0 + S_b(t) u_1 + \\ & \int_0^t S_b(t-s) N(v,\phi_b)(s) ds + \phi_0^b \int_t^\infty < \phi_0^b, N(v,\phi_b)(s) > ds + \\ & cos(t\sqrt{H_b})(O(|1-a| < r >^{-3})), \end{aligned} \tag{6.159}$$

$n_{-,w}(0) = O(\delta)$ being otherwise arbitrary.

Applying now exactly the same procedure as the one we used to prove the Cauchy property of the sequences $(a_i)_{i\geq 0}$ and $(u_i)_{i\geq 0}$ in Proposition 6.9 one can show that the map Φ satisfies the following Lemma with $A = \Phi$, $S = (1-\delta, 1+\delta) \times \mathscr{T}$ and $T = \mathscr{B}_{1,\delta} \times \mathscr{B}_{2,\delta}$, S and T being equipped with the metrics relative to the metric $d_{\mathscr{U}}$ defined in (6.155).

Lemma 6.13. *Let S be a complete metric space and T an arbitrary metric space. Suppose that $A : S \times T \to S$ is such that there exists $0 < \gamma < 1$ with*

$$\begin{aligned} sup_{t\in T} d_S(A(x,t), A(y,t)) &\leq \gamma d_S(x,y)\ ,\ x,y \in S \\ sup_{x\in S} d_S(A(x,t_1), A(x,t_2)) &\leq C_0 d_T(t_1,t_2)\ ,\ t_1,t_2 \in T. \end{aligned} \tag{6.160}$$

Then for every $t \in T$ there exists a fixed point $x(t) \in S$. Moreover these points satisfiy the bounds

$$d_S(x(t_1), x(t_2)) \leq \frac{C_0}{1-\gamma} d_T(t_1,t_2)\ ,\ t_1,t_2 \in T.$$

Proof. See [9] (Lemma 8, p.20) for a complete proof. □

The proof of the Lipschitz continuity of h relative to (f_0, u_1) is then completed by considering (6.156).

Proof of Theorem 6.1

We conclude this chapter by proving that our solution u scatters to free wave when $t \to \infty$ which will therefore conclude the proof of the Theorem 6.1.

End of the proof of Theorem 6.1: Collecting the results obtained in Propositions 6.3 and 6.9 one has proved the existence of a solution u of the linearized wave equation satisfying the free time decay dispersion.

The discussion of Section 6.4 identified a stable manifold of starting Cauchy values under the form of a Lipschitz graph as described in the statement of Theorem 6.1.

Coming back to the ODE notation as in (6.5) the goal is to show that there exists Cauchy values (f_1, f_2) in some well-chosen Hilbert space such that

$$U(t) = U_0(t) + o(1)\ ,\ t \to \infty, \tag{6.161}$$

where U_0 is the free-wave solution with Cauchy values (f_1, f_2).

A natural candidate for the Hilbert space in concern is $\mathcal{E} := \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

The norm for the space $\mathcal{E}$ is the *energy norm* given by

$$\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{\mathcal{E}}^2 := \|\nabla u_1\|_2^2 + \|u_2\|_2^2. \tag{6.162}$$

We shall soon see that this natural norm is not the most efficient to perform calculation the reason being that the operator in concern is not the Laplacian $-\triangle$ but the linearized operator H_∞.

One then wants to show that

$$\|U(t) - U_0(t)\|_{\mathcal{E}} \to 0\,,\ t \to \infty. \tag{6.163}$$

Because the free Hamiltonian has only essential spectrum $[0, \infty)$, considering the decay in time property of $n_+(t)$ and $n_-(t)$, it suffices to show that

$$\|\tilde{U}(t) - U_0(t)\|_{\mathcal{E}} \to 0\,,\ t \to \infty. \tag{6.164}$$

Moreover if one takes into account the asymptotic completeness of H_∞ (see [10] or [13]), he is left with finding $(\tilde{f}_1, \tilde{f}_2) \in P_e\mathcal{E}$ such that

$$\|\tilde{U}(t) - e^{tJ\mathcal{H}_\infty}(\tilde{f}_1, \tilde{f}_2)^T\|_{\mathcal{E}} \to 0\,,\ t \to \infty, \tag{6.165}$$

where $(.,.)^T$ stands for transposition.

The idea is now to find some other Hilbert space where one will be able to perform the most effective calculations.
The space in question should be such that the propagator group $A := \{e^{tJ\mathcal{H}_\infty}\ :\ t \geq 0\}$ is unitary.
The natural candidate to start with is $\mathcal{E}_\infty$ defined by $\mathcal{E}_\infty := \dot{H}^1 \times L^2$ with norm

$$\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{\mathcal{E}_\infty}^2 := \|\sqrt{H_\infty} u_1\|_2^2 + \|u_2\|_2^2. \tag{6.166}$$

We shall not give all the calculation details concerning the proof of unitarity of A.
One only mentions that the principal reason why it works is that one will have to deal with

operators of the following types

$$\begin{aligned}\sqrt{H_\infty}cos(t\sqrt{H_\infty}) &= cos(t\sqrt{H_\infty})\sqrt{H_\infty}\\ \sqrt{H_\infty}sin(t\sqrt{H_\infty}) &= sin(t\sqrt{H_\infty})\sqrt{H_\infty}\\ \sqrt{H_\infty}\frac{sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} &= sin(t\sqrt{H_\infty}),\end{aligned} \tag{6.167}$$

whose commutation between them is verified regarding the domains
$D(cos(t\sqrt{H_\infty})) = D(sin(t\sqrt{H_\infty})) = L^2$.

Now considering the linear dispersive estimates for $\tilde{u}$ and for the first component of the linearized propagaor $e^{tJ\mathcal{H}_\infty}(f,g)^T$, it suffices, by (6.35), to show that

$$\|e^{-tJ\mathcal{H}_\infty}\tilde{U}(t) - (\tilde{f}_1,\tilde{f}_2)^T\|_{\mathcal{E}_\infty} \to 0\ ,\ t\to\infty, \tag{6.168}$$

using the unitarity of A.

By (6.16) one writes

$$e^{-tJ\mathcal{H}_\infty}\tilde{U} = \tilde{U}(0) + \int_0^t e^{-sJ\mathcal{H}_\infty}P_e W(s)ds, \tag{6.169}$$

from which it follows that if such Cauchy values $(\tilde{f}_1,\tilde{f}_2)$ are existing they must have the form

$$\begin{pmatrix}\tilde{f}_1\\ \tilde{f}_2\end{pmatrix} = \tilde{U}(0) + \int_0^\infty e^{-sJ\mathcal{H}_\infty}P_e W(s)ds. \tag{6.170}$$

They are clearly in $P_e\mathcal{E}$ and are well-defined considering the form of $W(s)$, the boundedness of $cos(t\sqrt{H_\infty})_{2\to 2}$ and $sin(t\sqrt{H_\infty})_{2\to 2}$ together with the norm controls (6.53) applied to u. This is the place, for precisely obtaining an absolut convergent integral in the definition of $(\tilde{f}_1,\tilde{f}_2)$ given in (6.170), where one remarks the reason for having asked for a wave u with the free time decay.

By an analog argumentation they satisfy (6.168).

This ends the proof of the theorem 6.1. □

7 Concluding Remarks

For concluding the text one first remarks that the result obtained in [9] is improved in the sense that one has not to impose the compact support assumption for the couple of Cauchy starting values (f_0, u_1). The price to pay is the control of a weighted Sobolev norm as in (6.54). Another point is that the iterative method used in this context, for obtaining the stable manifold, makes the calculation substantially easier than in [9], avoiding the construction of the topology needed for obtaining the fixed point argument.

Further research has been considered when one is willing to generalize the present construction applied to higher dimension $(n \geq 4)$. Not all the details have for the time being been established but the main technical points are under control for being able to apply the same kind of considerations than the ones used in the case of dimension 3.

The deepest issue to consider is the possibility to apply the Distorted Fourier representation for obtaining the linear dispersive estimates in higher dimensional cases.
When one is writing the linearized operator in dimension one, starting with dimensions $n \geq 4$, he obtains the following Sturm-Liouville expression

$$\mathscr{L} = -\frac{d^2}{dr^2} + V(r) + \frac{c(n)}{r^2},$$

with $V(r)$ being a smooth, L^∞, potential on $[0, \infty)$ and $c(n)$ some constant only depending on the dimension.
One thus observes that the potential has now a singular part near the boundary point 0 making 0 in the limit point case. Being in the limit point case at 0 and ∞, Theorem 3.13 tells one that $\mathscr{L}$ is self-adjoint. This has the advantage that one does not need to consider extensions.
The diificulty coming from self-adjointness is that an L^2 solution $\tilde{\psi}_+$, satisfying then $(\mathscr{L} - z)\tilde{\psi}_+$, $z \in \mathbb{C} \setminus \mathbb{R}$, is not anymore existing avoiding the easy construction of the Green function as in (3.19).

The solution to this difficulty is to consider the restriction of $\mathscr{L}$ to subintervals $(0, x_0]$ and $[x_0, \infty)$ with a ghost variable x_0, essentially needed to tract the regular behaviour of the potential part $V(r) + \frac{c(n)}{r^2}$ of $\mathscr{L}$ at this point x_0.

One is therefore able to consider $\mathscr{L}_{(0,x_0]}$ and $\mathscr{L}_{[x_0,\infty)}$ in the same way than in the case of dimension 3, getting the usual fundamental system of solutions $\{\tilde{\phi}(., z, x_0), \tilde{\theta}(., z, x_0)\}$ with prescribed boundary values at x_0. One thus obtains the existence of a unique $L^2([x_0, \infty), dr)$-solution $\tilde{\psi}_+(., z, x_0)$ (which is, by unicity, L^2 on any interval of the form $[b, \infty)$ with $b > 0$) satisfying $\tilde{\psi}_+(x_0, z, x_0) = 1$ which can thus be expressed in the form

$$\tilde{\psi}_+(., z, x_0) = \tilde{\theta}(., z, x_0)) + \tilde{m}(z, x_0)\tilde{\phi}(., z, x_0).$$

The remaining two technical problems are the presence of this artificial x_0 which has to be removed in a raisonnable way and the fact that $\tilde{m}(z, x_0)$ is not a priori an Herglotz function, therefore not being a priori able to invert an integral representation for $\tilde{m}$ to obtain some regular Borel measure. One can in fact prove that $\tilde{m}$ does not depend on x_0 and that it shows enough properties similar to the ones satisfied by an Herglotz function such that one is still able to construct a Borel measure from it.
The required Distorted Fourier representation in the present case follows then in a similar way than in dimension $n = 3$.

The Fourier basis in this context is not anymore the fundamental solution $\tilde{\phi}$ as in dimension 3 but a specific conjectured solution ϕ (which can be shown to exist) satisfying some very constraining hypotheses established such that the whole story can work.

One also notes that, in dimensions $n \geq 5$, the function which was resonance in dimension 3 becomes degeneracy, being now in L^2.
One has therefore to project orthogonally to the latter for being able to get the free dispersive decay of the wave. This is performed considering writing the Fourier basis in a similar way than seen in (4.8) extracting the degeneracy part for being able to projext orthogonally of it.

Apart from those technical difficulties the method applied in the present context for constructing the stable manifold works well, being even easier to apply essentially because one does not need to consider any resonance part anymore (not decaying in time), even in dimension 4. The latter case is due to higher free dispersive decay when one is increasing the dimension.

Another direction to explore is the way of obtaining similar results than in the present case but without the radial hypothesis. Here, already in dimension $n = 3$, the linearized operator has a

degeneracy, which is the gradient of the Aubin-Talenti solution.
Therefore all the spectral analysis would have to be reconsidered.
This is certainly a more difficult problem to solve than in the present context and one would certainly have to consider the Lorentz group, which is the symmetry group for the wave equation, to reduce the complexity of the problem.

Many other directions can in fact certainly been surveyed, for example when one is willing to look for the nature of the blowups solutions when leaving the stable manifold, using, when appropriate, more advanced analytical techniques such as the Microlocal analysis and the Concentration Compactness principle.

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