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STABILIZED REDUCED BASIS METHOD FOR PARAMETRIZED SCALAR ADVECTION–DIFFUSION PROBLEMS AT HIGHER PÉCLET NUMBER: ROLES OF THE BOUNDARY LAYERS AND INNER FRONTS

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Abstract. Advection–dominated problems, which arise in many engineering situations, often require a fast and reliable approximation of the solution given some parameters as inputs. In this work we want to investigate the coupling of the reduced basis method – which guarantees rapidity and reliability – with some classical stabilization techiques to deal with the advection–dominated condition. We provide a numerical extension of the results presented in [1], focusing in particular on problems with curved boundary layers and inner fronts whose direction depends on the parameter.

1 INTRODUCTION

Advection–diffusion equations are widely used to model physical phenomena. For instance, we recall heat transfer phenomena [2] and the diffusion of pollutants in the atmosphere or waters [3, 4]. These equations may often depend on several parameters involving both the coefficients, e.g. the thermal conductivity or the wind (transport) direction, and the domain itself of the equation, e.g. the geometry of a thermal cooling system. In engineering applications, accurate numerical approximations of the solution are needed and these can be provided, for instance, by the Finite Element (FE) method [5]. In some situations, however, a very fast approximation of the solution is needed and a complete FE simulation may not suit one's purpose, because it can turn out to be excessively time-demanding. In particular, when dealing with *real–time* simulations or *many–query* situations (i.e., the same equation must be solved for many different parameters values), it is possible to resort to the Reduced Basis (RB) method [6], which aims to recover a *high–fidelity* approximated solution, often called *truth* solution, by performing a Galerkin projection onto a low–dimensional space. Moreover, the RB method provides rigorous *a posteriori* error estimators which guarantees the reliability of the method. The computational efficiency of the RB method is reached thanks to the fact that it is split in two computational stages. During the first expensive stage, to be performed once, a low–dimensional space – spanned by FE solutions computed at properly chosen parameter values – is built and stored, together with all the structure needed to assemble and solve the small linear problem associated with the RB Galerkin projection. The assembly and the solution of this small linear problem, which provides the fast approximation of the solution given a value of the parameters, is called Online stage. The latter is computationally inexpensive and can be repeated an arbitrary number of times.

The RB method has been successfully applied to the approximation of advection–diffusion problems in [2, 7, 8, 9, 10]. These cited works deal with problems in which the Péclet number, which is the ratio between the advection term and the diffusion one, takes low values. More precisely, the values taken by Péclet number are sufficiently small to avoid the numerical instabilities well known in literature [11].

It is possible to find many methods to fix the instability problems. A widely studied class of them is represented by the strongly consistent stabilization methods [11], which we will briefly recall in Section 2. First results about the coupling of stabilization methods and RB method can be found in [3, 4]. Recently, new methods have been proposed as regards the Model Order Reduction for advection dominated problems. We recall [12] in which a Petrov–Galerkin approach involving a double Greedy strategy is followed. Moreover, in [13] a method based on Variational Multiscale (VMS) and Proper Orthogonal Decomposition (POD) is proposed. This method has also been extended to Navier–Stokes equation in [14].

The present work is a numerical extension of the results presented in [1]. In the latter, a comparison between two possible computational–split stabilization strategies is carried out. The first strategy, called *Offline–Online stabilized* method, consists in performing both the Offline and the Online stages with respect to a stabilized problem. This strategy fits the standard RB theory presented in [6] and it was actually used in [3, 4]. In the second strategy, called *Offline–only stabilized* method, only the Offline stage is performed using the stabilized form, while the Online Galerkin projection is done using the standard advection diffusion form. To motivate this approach we observe that the RB solution is actually a linear combination of precomputed particular solutions of the equations, so that it can be reasonable to expect that if the basis functions are stable, so it is the RB solution. In [1] it is shown that this is not valid in general. However, if we are using a Streamline Upwind Petrov Galerkin (SUPG) stabilization, the *Offline–only stabilized* method can produce still stable solutions, especially when the boundary layers are parallel to the transport field.

In this work we want to extend the study carried out in [1] with new investigations, by testing the proposed methods on geometries with curved boundaries, which imply the presence of curved boundary layers. Moreover, we perform tests with multiple inner fronts whose direction depends on the parameter and with immersed bodies in the thermal flow.

In Section 2 we briefly recall the methods used, while in Section 3 we show the numerical results. Conclusions follow.

2 OVERVIEW OF THE STABILIZED REDUCED BASIS METHOD

We follow the framework presented in [1]. Let the domain Ω be an open subset of $\mathbb{R}^2$, and let the parameter space $\mathcal{D}$ be a subset of $\mathbb{R}^P$, where P is the number of parameters. Given a value $\boldsymbol{\mu} \in \mathcal{D}$, the problem we are going to consider is

$$-\varepsilon(\boldsymbol{\mu})\Delta u(\boldsymbol{\mu}) + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla u(\boldsymbol{\mu}) = h(\boldsymbol{\mu}) \quad \text{on } \Omega. \tag{1}$$

with Dirichlet, Neumann or mixed boundary conditions. The diffusion coefficient $\varepsilon(\boldsymbol{\mu})$ and the advection field $\boldsymbol{\beta}(\boldsymbol{\mu})$ are sufficiently regular functions defined on Ω with values in $(0, +\infty)$ and $\mathbb{R}^2$, respectively. The source term $h(\boldsymbol{\mu})$ is a L^2 function defined on Ω. From the general advection–diffusion PDE theory (see e.g. [15]), in order to ensure the well posedness of problem (1) we must make proper assumptions on the coefficients $\varepsilon(\boldsymbol{\mu})$ and $\boldsymbol{\beta}(\boldsymbol{\mu})$. In our numerical tests, for each value of the parameter $\boldsymbol{\mu} \in \mathcal{D}$, we will assume that it exists $\bar{\varepsilon} > 0$ such that $\varepsilon(\boldsymbol{\mu}) \geq \bar{\varepsilon}$ and the advection field $\boldsymbol{\beta}(\boldsymbol{\mu})$ is directed outwards Ω on the Neumann boundaries, i.e., Neumann conditions are only imposed on outflow boundaries. The associated problem in variational form is

$$\text{find } u(\boldsymbol{\mu}) \in H_0^1(\Omega) \text{ such that:} \quad a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}) \quad \forall\, v \in H_0^1(\Omega), \tag{2}$$

where a is the bilinear form associated with the advection–diffusion operator, while the right–hand side functional $f(\cdot; \boldsymbol{\mu})$ depends on the source term $h(\boldsymbol{\mu})$ and on the boundary conditions.

When the advection term dominates the diffusion one, the numerical approximation of problems like (1) can be difficult [11]. In particular, spurious oscillations in the numerical solution may arise. More precisely, when using a piecewise–linear FE approximation defined on a triangulation $\mathcal{T}^{\mathcal{N}}$ of the domain Ω, we say that a problem is *advection–dominated in* $K \in \mathcal{T}^{\mathcal{N}}$ when the following condition holds:

$$\mathbb{P}\mathsf{e}_K(\boldsymbol{\mu})(x) := \frac{|\boldsymbol{\beta}(\boldsymbol{\mu})(x)| h_K}{2\varepsilon(\boldsymbol{\mu})(x)} > 1 \quad \forall \boldsymbol{x} \in K \quad \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{3}$$

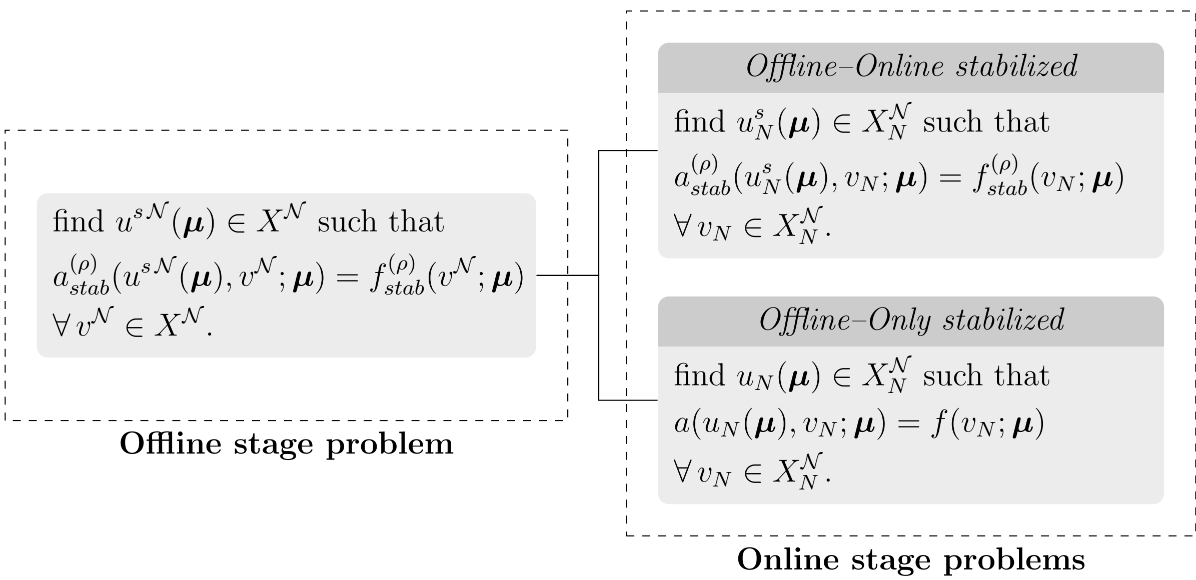
having denoted with h_K the diameter of K.

Many strategies have been proposed to deal with the FE approximation of advection dominated problems. An example can be the strongly consistent stabilization methods [5, 11]. The basic idea is to modify the advection–diffusion bilinear form a by adding a stabilization term to the standard form. Denoting with $X^{\mathcal{N}}$ the piecewise linear FE space on $\mathcal{T}^N$, given a value ρ, the general form of these methods is

$$a_{stab}^{(\rho)}(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = a(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) + \sum_{K \in \mathcal{T}_h} \delta_K \int_K L^{\boldsymbol{\mu}} w^{\mathcal{N}} \left(\frac{h_K}{|\boldsymbol{\beta}(\boldsymbol{\mu})|} (L_{SS}^{\boldsymbol{\mu}} + \rho L_S^{\boldsymbol{\mu}}) v^{\mathcal{N}} \right), \quad (4)$$

for each $v^{\mathcal{N}}, w^{\mathcal{N}} \in X^{\mathcal{N}}$, where $L^{\boldsymbol{\mu}}$ is the advection–diffusion operator, while L^S and L^{SS} are its symmetric and skew–symmetric parts, respectively. Also the right–hand side of the equation must be modified in order to recover the strong consistency and we will denote in with $f_{stab}^{(\rho)}(\cdot; \boldsymbol{\mu})$. For $\rho = 0$ we have the so called Streamline Upwind Petrov–Galerkin (SUPG) method, for $\rho = 1$ the Galerkin Least–Squares (GLS) method and for $\rho = -1$ the Douglas–Wang (DW) method. We observe that, when using piecewise linear FE, the three mentioned stabilization techniques actually coincide. We refer to [11] and the references therein for a detailed presentation and analysis of the strongly consistent stabilization methods.

In order to couple the FE stabilization techniques with the RB machinery, we resort to the *Offline–Online stabilized* and the *Offline–only stabilized* strategies introduced and studied in [1]. In this framework, to compute the particular solutions which spans the reduced space $X_N^{\mathcal{N}}$, we perform the Offline stage of both methods with respect to the same stabilized bilinear form (4). The differences between the two stabilization strategies arise during the Online stage, in which different Galerkin projections are exploited.



In order to carry out succesfully a RB approach, we need to make proper assumptions on the forms involved. In particular we assume the *affine dependence* on the parameter [6], i.e.,

$$a(w^{\mathcal{N}}, v^{\mathcal{N}}; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu}) a^q(w^{\mathcal{N}}, v^{\mathcal{N}}) \quad \forall \boldsymbol{\mu} \in \mathcal{D}. \quad (5)$$

where Θ_a^q, $q = 1, \ldots, Q_a$, are $\mathcal{D} \to \mathbb{R}$ functions, while a^q, $q = 1, \ldots, Q_a$, are parameter independent bilinear forms on $X^{\mathcal{N}}$. We make this assumption on all the forms involved.

The following upper bound can be proven (see [1]).

Proposition 1 *If $\rho = 0$ (SUPG), the following estimate of the error between the Offline-only stabilized approximation $u_N(\boldsymbol{\mu})$ and the stabilized FE approximation $u_N^{s,\mathcal{N}}(\boldsymbol{\mu})$ holds:*

$$|||u_N(\boldsymbol{\mu}) - u_N^{s,\mathcal{N}}(\boldsymbol{\mu})|||_{\boldsymbol{\mu}} \leq \; |||u_N^s(\boldsymbol{\mu}) - u_N^{s,\mathcal{N}}(\boldsymbol{\mu})||| + h\, C(\boldsymbol{\mu}) \|\boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla(u_N^s(\boldsymbol{\mu}) + g_h)\|_{L^2(\Omega)} \quad (6)$$

where $u_N^s(\boldsymbol{\mu})$ is the Offline–Online stabilized solution, g_h is the lifting of the Dirichlet boundary condition, $|||\cdot|||_{\boldsymbol{\mu}}$ is the norm induced by the symmetric part of the bilinear form $a(\cdot, \cdot; \boldsymbol{\mu})$ and $C(\boldsymbol{\mu})$ is such that $|v|_{H^1(\Omega)} \leq C(\boldsymbol{\mu}) |||v|||_{\boldsymbol{\mu}}$, for all $v \in H_0^1(\Omega)$. The value h is the maximum element diameter of the mesh $\mathcal{T}^{\mathcal{N}}$.

For the sake of presentation, we decided to set our problem on a parameter independent domain Ω. However, it is possible to consider domains which are the transformation of a reference domain through suitable mappings [1, 6, 9].

3 NUMERICAL RESULTS

In this section we show some numerical results obtained with the model problem we considered. In Subsection 3.1 we test the presented methodology on a problem which can model the diffusion of some quantity transported by a fluid flowing in a stenosed vessel. In this case we tested also different strongly consistent stabilization techniques. In Subsection 3.2 we focus on a system made by three bars immersed in a flowing fluid in which they release a particular substance (e.g. pollutant, oxygen, drugs, etc.).

3.1 Stenosed vessel test case

We consider an advection–diffusion problem defined on the stenosed vessel sketched in Figure 1. The problem is

$$\begin{aligned}
-\frac{1}{\mu_1}\Delta u + \boldsymbol{\beta}(\mu_2) \cdot \nabla u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \\
\frac{1}{\mu_1}\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_3, \\
\frac{1}{\mu_1}\frac{\partial u}{\partial n} &= 1 \quad \text{on } \Gamma_4.
\end{aligned} \quad (7)$$

where the divergence free advection field $\boldsymbol{\beta}(\mu_2)$ is defined as follows:

$$\boldsymbol{\beta}(x, y; \mu_2) = -\frac{16}{25} y \left(y - \frac{5}{2} \right) \mathbf{i} + \mu_2 \, \mathbf{j}, \quad (8)$$

having denoted with $\mathbf{i}$, $\mathbf{j}$ the usual unit vectors of $\mathbb{R}^2$.

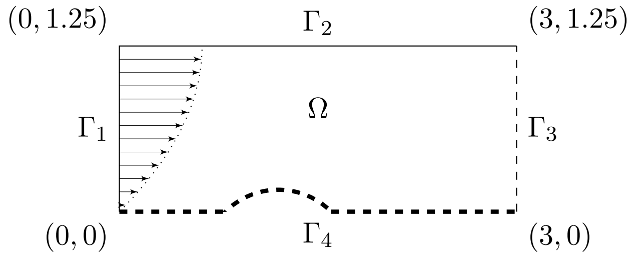
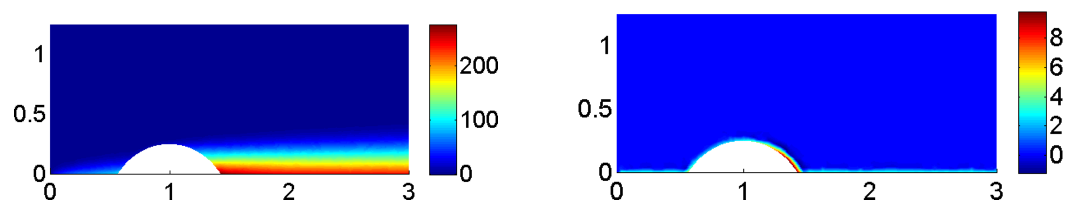


Figure 1: Stenosed vessel test case. Boundary conditions: Neumann condition $\frac{1}{\mu_1}\frac{\partial u}{\partial n} = 1$ on the bold dashed side, homogeneous Neumann on the dashed side and homogeneous Dirichlet conditions on the remaining sides.



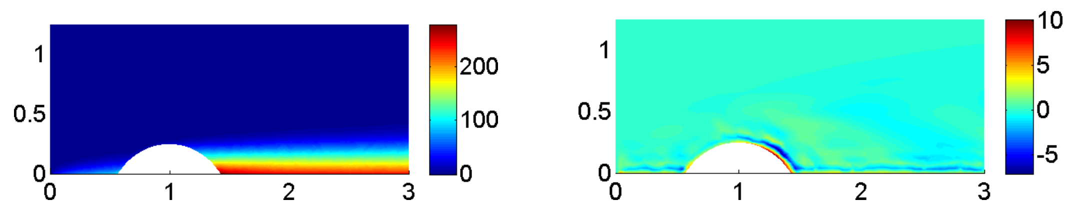
(a) $\boldsymbol{\mu} = (800, 0)$ (b) $\boldsymbol{\mu} = (800, -0.3)$

Figure 2: Stenosed vessel test case. Representative *Offline–Online stabilized* approximated solutions.

We chose the SUPG stabilization technique ($\rho = 1$) and we let the parameter $\boldsymbol{\mu} = (\mu_1, \mu_2)$ in the parameter space $\mathcal{D} = [100, 1000] \times [-0.5, 0]$. The Greedy algorithm produced 61 basis, while the FE space dimension is 2341. In Figures 2 and 3 we show some relevant RB solutions obtained with both the stabilization strategies, while in Figure 4 we compare the approximation errors. We observe that, when the advection field is almost parallel to the boundary layer (i.e., when $\boldsymbol{\mu}_2 = 0$), the *Offline–only stabilized* solution does not show instabilities. This is in accordance with the results in [1].

3.1.1 Test with different stabilization techniques

We tried also to perform a RB approximation of problem (7) comparing the results obtained using either SUPG or GLS stabilization. We chose to use piecewise quadratic



(a) $\boldsymbol{\mu} = (800, 0)$ (b) $\boldsymbol{\mu} = (800, -0.3)$

Figure 3: Stenosed vessel test case. Representative *Offline–Only stabilized* approximated solutions.

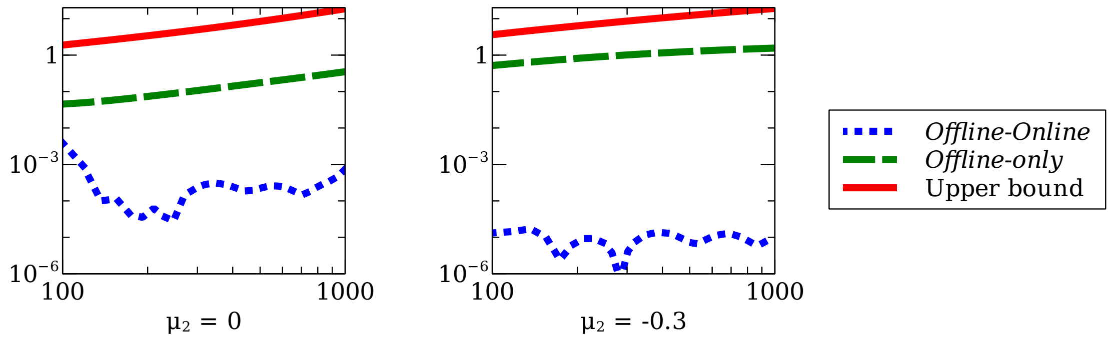
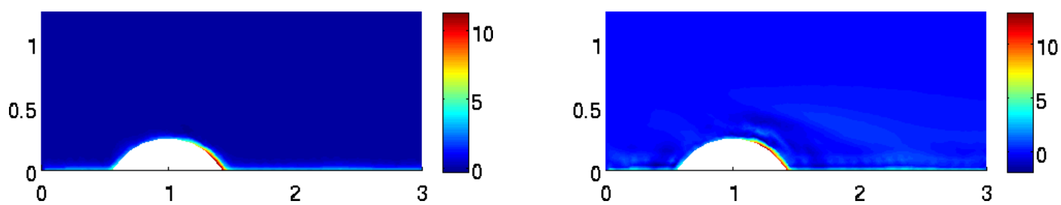


Figure 4: Stenosed vessel test case. Approximation error and error *a priori* upper bound of Proposition 1. The curves represent the error as a function of μ_1, given a fixed value of μ_2.



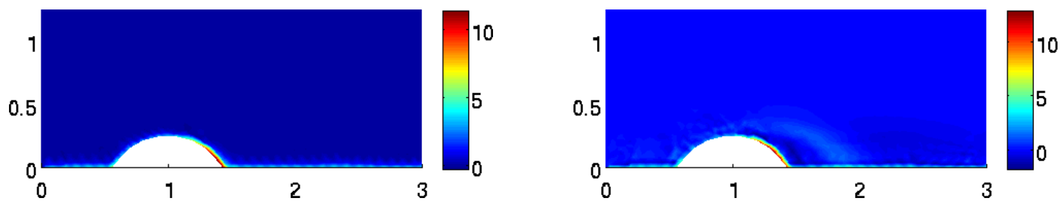
(a) *Offline–Online stabilized* (b) *Offline–Only stabilized*

Figure 5: Stenosed vessel test case. RB solutions obtained using SUPG.

finite elements, as done e.g. in [1]. In Table 1 we show some data about the computation. Here $\mathcal{N}$ is the dimension of the *truth* FE space while N the dimension of the RB space. We reported also the number of iteration performed by the SCM algorithm [6] used to allow the approximation of the $\boldsymbol{\mu}$–dependent lower bound of the coercivity constant. We observe that the GLS method requires a higher number of SCM iterations. In Figure 5 we show the solution obtained using the SUPG stabilization coupled with the *Offline–Online* and *Offline–only* strategies. We observe that the *Offline–only stabilized* solution shows notable instability phenomena. The situation of the GLS computation is similar.

Stabilization	$\mathcal{D}$	$\mathcal{N}$	N	SCM iter.
SUPG	$[100, 1000] \times [-0.3, -0.25]$	2318	19	10
GLS			23	23

Table 1: Stenosed vessel test case. Comparison of SUPG and GALS stabilization.



(a) *Offline–Online stabilized* (b) *Offline–Only stabilized*

Figure 6: Stenosed vessel test case. RB solutions obtained using GLS.

3.2 Bars test case

We consider an advection–diffusion problem defined on the domain Ω sketched in Figure 1. The problem is

$$\begin{aligned}
-\frac{1}{\mu_1}\Delta u + \boldsymbol{\beta}(\mu_2)\cdot\nabla u &= 0 \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_4,\\
u &= 1 \quad \text{on } \Gamma_5 \cup \Gamma_6 \cup \Gamma_7,\\
\frac{1}{\mu_1}\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_3.
\end{aligned} \tag{9}$$

where the divergence–free advection field $\boldsymbol{\beta}(\mu_2)$ is defined as follows:

$$\boldsymbol{\beta}(x, y; \mu_2) = \left[-\frac{64}{49}y\left(y - \frac{7}{4}\right)\right]\mathbf{i} + \mu_2\,\mathbf{j}, \tag{10}$$

having denoted with $\mathbf{i}$, $\mathbf{j}$ the usual unit vectors of $\mathbb{R}^2$. The x–component of the advection field has a parabolic profile and it is null along Γ_1 and Γ_3. Its maximum value is 1. We point out that Γ_5, Γ_6 and Γ_7 are circles of radius 0.5 and are centered in $(0.75, 0.5)$, $(0.75, 1.25)$, $(1.75, 0.875)$, respectively.

In our test, we let the parameter $\boldsymbol{\mu} = (\mu_1, \mu_2)$ in the parameter space $\mathcal{D} = [10, 100] \times [-0.5, 0.5]$. The FE *truth* space dimension id 2407 and the Greedy algorithm produced 125 basis. In Figures 8 we show some relevant RB solution obtained with both *Offline–Online* and *Offline–only stablized* method. We observe that in this case the performances of the *Offline–Online stabilized method* are significantly better. In Figure 9 we compare the approximation errors. As already pointed out in [1], when dealing with a problem characterized by internal layers whose directions depend on the parameter, it is required a conspicuous number of reduced basis functions in order to properly resolve the layer during the Online stage.

4 CONCLUSIONS

We have compared numerically two possible stabilization strategies, the *Offline–Online stabilized* and the *Offline–only stabilized*, on domains with curved boundary and in presence of multiple internal layers. In general, the *Offline–Online stabilized* gives better

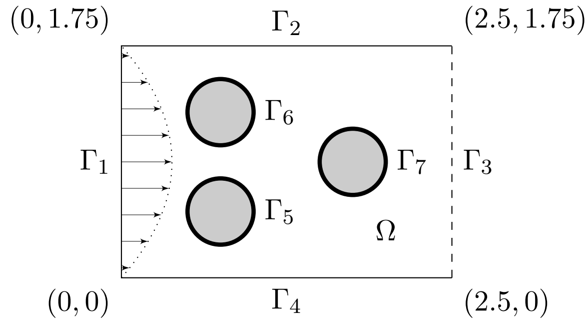
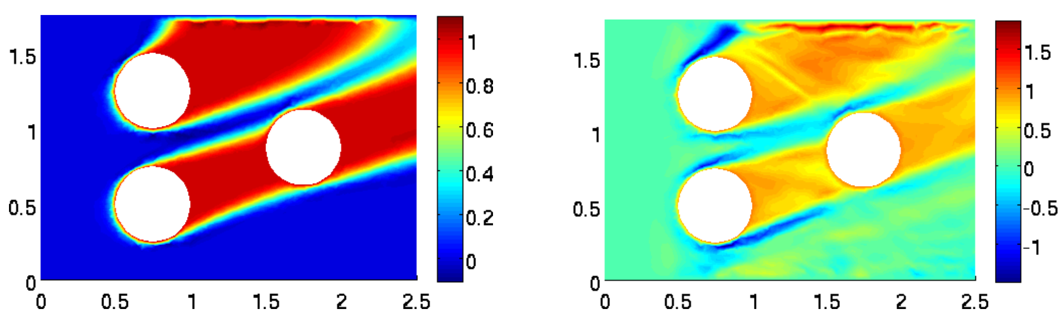


Figure 7: Bars test case domain. Boundary conditions: Dirichlet condition $u = 1$ on the bold sides, homogeneous Neumann on the dashed side, homogeneous Dirichlet on the remaining sides. The grey disks are not part of the domain.



(a) *Offline–Online stabilized*

(b) *Offline–Only stabilized*

Figure 8: Stenosed vessel test case. RB solutions obtained using GLS.

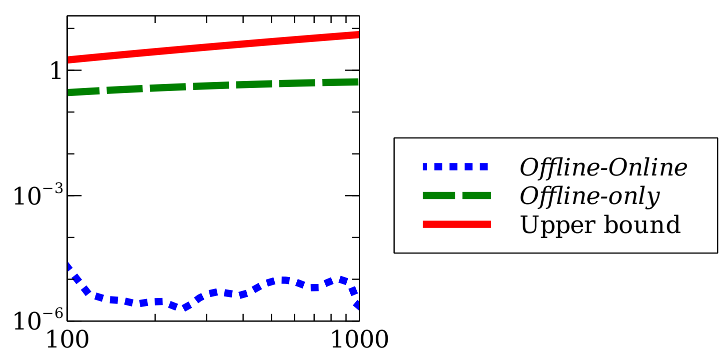


Figure 9: Bars test case. Approximation error and error *a priori* upper bound of Proposition 1. The curves represent the error as a function of μ_1, given $\mu_2 = 0$.

results, while *Offline–only stabilized* solution can often show instability phenomena which mainly arise when the advection field is not parallel to the boundary layer. Moreover, as observed in [1], the *Offline–Online stabilized* method can also be provided with the standard RB *a posteriori* error estimators.

Possible extensions of this work could involve the reduced basis approximation of a scalar advection–diffusion problem in which the advection field is the velocity of a fluid, which has to be computed according to the parameter. Future research will be then devoted to stabilization strategies for vectorial and non–linear problems (e.g. Navier–Stokes flows).

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