# Probabilistic analysis of two models of ideal memristor with external noise


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***Abstract*—This work is devoted to probabilistic analysis of two models of the ideal memristor with an external Gaussian noise. First we study the charge-controlled ideal memristor and analyze how the applied voltage in the form of a stationary Gaussian noise influences the probability density function of the charge and resistance. For comparison we pay attention to the case of applied current in the same form. Further the current-controlled ideal memristor with the external fluctuations in the form of a stationary Gaussian noise is under consideration. The exact results reported are based on well-known theorems of the probability theory.**



***Index Terms*—ideal memristor, charge-controlled memristor, probability density function, external Gaussian noise**


## I. Introduction

The memristor is a new fundamental element of the electrical circuit that dissipates energy and has memory. It was theoretically predicted by Chua in 1971 [1], but found its hardware implementation only in 2008 [2]. According to the axiomatic definition of the ideal memristor [1], its properties are unambiguously determined by the memristance versus charge map. The original model of the "HP memristor" [2] introduces this map via a linear function that represents this memristor as a variable resistor whose resistance is linearly dependent on the amount of charge flowing through. However, a lot of subsequent experimental works in this area, the current-voltage characteristics and various physical considerations have indicated the need to use nonlinear models to correctly describe the properties of memristors, e.g. a nonlinear dependence of the resistance on an external controlling variable. Various nonlinear dependencies of the resistor on charge were proposed [3], [4], and attempts were done to construct the nonlinear characteristics by probing with special deterministic perturbations [5], [6].

Currently, much attention is focused on the study of the memristor behavior when applying to it external deterministic or random signals and its switching mechanism [7]–[10], whose resistance varies according to the voltage applied to them, or the current flowing through (see, for example, reviews [11]–[14]). Unfortunately, at the moment theoretical, and even more experimental papers on the study of the statistical properties of the memristor as a separate element of the electrical circuit is not enough. There are only a few studies of the steady-state probability distributions of the current flowing through the memristor in a circuit with a resistance or a capacitor, when a white Gaussian noise source is applied to it [8], [9].

In this paper we demonstrate how the Gaussian fluctuations of the applied voltage or current may change the memristor's state. These changes are indicated on an example of the ideal memristor using the probability characteristics of its resistance.

## II. General formulas

The simplest model of a memory device is the ideal memristor proposed in [1]. For this nonlinear element of electrical circuit, the Ohm's law and the associated equations of state are written as follows

$$U(t) = R(q)I(t), \quad I(t) = \frac{dq}{dt}, \tag{1}$$

where $U(t)$, $I(t)$ and $q(t)$ are the voltage, the current and the charge on the memristor respectively.

As follows from (1), the ideal memristor is an integrable model and, hence, can be defined by an equivalent algebraic function

$$w(t) = \int_0^t U(t)dt = \int_0^{q(t)} R(q)dq = \Phi(q(t)). \tag{2}$$

Let us apply to memristor the voltage $U(t)$ in the form of a stationary Gaussian noise with non-zero mean $U_0$ and the correlation function $K(\tau)$. According to (2), the random process $w(t)$ is again a Gaussian random process with the following probability distribution

$$P_w(y,t) = \frac{1}{\sqrt{4\pi D(t)}} \exp\left\{-\frac{(y-U_0 t)^2}{4D(t)}\right\}, \tag{3}$$

where

$$D(t) = \int_0^t (t-\tau)\, K(\tau) d\tau. \tag{4}$$

Then we can apply the theorem of the probability theory to calculate from (2)-(3) the probability density function (PDF) of the charge flowing through a memristor

$$P_q(z,t) = \frac{\Phi'(z)}{\sqrt{4\pi D(t)}} \exp\left\{-\frac{[\Phi(z)-U_0 t]^2}{4D(t)}\right\} \tag{5}$$

and as a consequence, the PDF of the resistance

$$P_R(r,t) = \frac{r}{\sqrt{4\pi D(t)}} \sum_k \frac{1}{|\Phi''(q_k(r))|} \times \exp\left\{-\frac{[\Phi(q_k(r)) - U_0 t]^2}{4D(t)}\right\}, \quad (6)$$

where $q_k(R)$ is the k-th branch of uniqueness of the function $R = \Phi'(q)$. By measuring of this PDF in the experiment, we can find an unknown algebraic function $\Phi(q)$ and determine all the statistical characteristics of the memristor which we are interested in.

## III. Charge-controlled memristor

In the case described in [3] the authors considered the following monotonic exponential dependence of the resistance on charge

$$R(q) = R_{ON} + \frac{\Delta R}{e^{-(q+q_1)/q_0} + 1}, \quad (7)$$

where $\Delta R = R_{OFF} - R_{ON}$ and $R_{ON} \ll R_{OFF}$. The parameter $q_0$ is a characteristic charge required to switch the memristor and specifies the steepness of the transition between low resistance state (LRS) $R_{ON}$ and high resistance state (HRS) $R_{OFF}$. The constant $q_1$ is a parameter determining the memristance at the initial moment of time $t = 0$.

The inverse to (7) monotonic function reads

$$q(R) = -q_1 - q_0 \ln\left(\frac{R_{OFF} - R}{R - R_{ON}}\right). \quad (8)$$

According to the definition (2) of the algebraic function, from (7) we find

$$\Phi(q) = qR_{ON} + q_0 \Delta R \ln\left(\frac{e^{(q+q_1)/q_0} + 1}{e^{q_1/q_0} + 1}\right), \quad (9)$$

$$\Phi''(q) = R'(q) = \frac{\Delta R}{q_0} \frac{e^{-(q+q_1)/q_0}}{\left(e^{-(q+q_1)/q_0} + 1\right)^2}. \quad (10)$$

Substituting (8)-(10) in (6), we obtain finally the exact expression for the PDF of the resistance $R(t)$ in the following form

$$P_R(r,t) = \frac{q_o \Delta R}{\sqrt{4\pi D(t)}} \frac{r}{(R_{OFF} - r)(r - R_{ON})} \times \exp\left\{-\frac{[\Phi(q(r)) - U_0 t]^2}{4D(t)}\right\}, \quad (11)$$

where

$$\Phi(q(r)) = -q_1 R_{ON} - q_0 \Delta R \ln\left(e^{q_1/q_0} + 1\right) + q_0 R_{OFF} \ln\left(\frac{\Delta R}{R_{OFF} - r}\right) - q_0 R_{ON} \ln\left(\frac{\Delta R}{r - R_{ON}}\right).$$

Let's consider two kinds of the applied random voltage. For the white Gaussian noise $U(t)$ with the correlation function

$$K(\tau) = 2D\delta(\tau), \quad (12)$$

where $2D$ is the noise intensity, we have to substitute $D(t) = Dt$ in (11) in accordance with (4). The plots of corresponding PDF of the resistance are depicted in Fig. 1. As seen from Fig. 1, the noise causes the memristor to switch

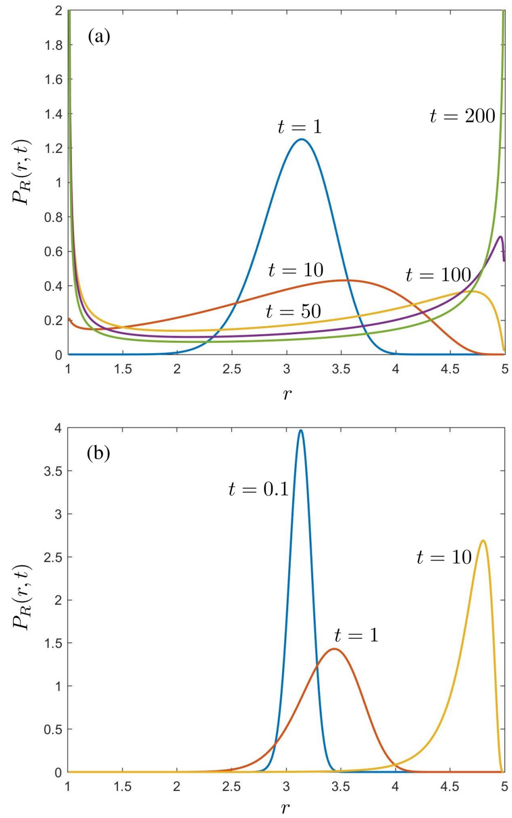


Fig. 1. Probability distribution of resistance (11) in the case of white Gaussian noise $U(t)$ for different time moments (a) $U_0 = 0$, (b) $U_0 = 1$. The parameters are $q_0 = 1$, $q_1 = 0.1$, $R_{ON} = 1$, $R_{OFF} = 5$, $D = 0.5$.

to high resistance state. In the case of the noise with zero mean one can observe both states.

For the case of colored Gaussian noise $U(t)$ with the exponential correlation function

$$K(\tau) = \sigma^2 \exp\left(-t/\tau_c\right), \quad (13)$$

where $\sigma^2$ and $\tau_c$ are the variance and the correlation time respectively, equation (4) gives

$$D(t) = \sigma^2 \tau_c \left[t - \tau_c\left(1 - e^{-t/\tau_c}\right)\right]. \quad (14)$$

The plots of corresponding PDF (11) of the memristance are shown in Fig. 2. In Fig. 2 we observe the same tendencies as in Fig. 1.

For comparison we apply to memristor the current $I(t)$ in the form of a stationary Gaussian noise with zero mean and the correlation function $K(\tau)$. In accordance with (1), the charge $q(t)$ is a Gaussian process with the following probability distribution

$$P_q(z,t) = \frac{1}{\sqrt{4\pi D(t)}} \exp\left\{-\frac{z^2}{4D(t)}\right\}. \quad (15)$$

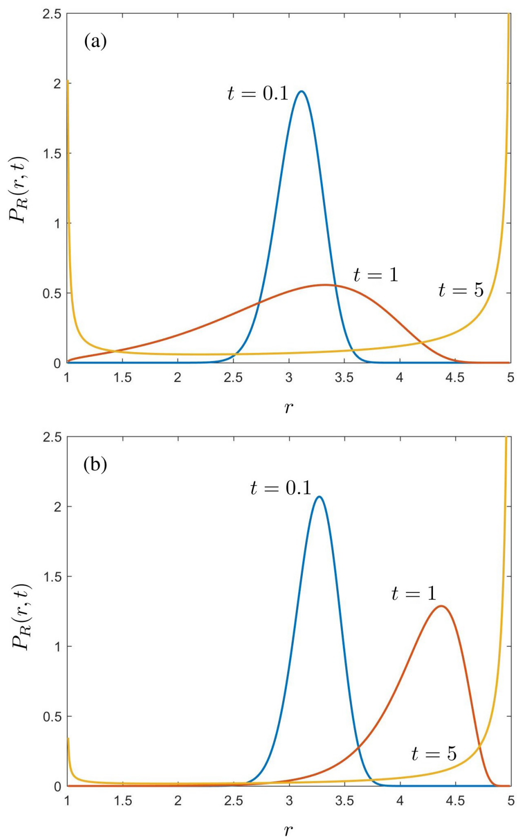


Fig. 2. Probability distribution of resistance (11) in the case of colored Gaussian noise $U(t)$ for different time moments (a) $U_0 = 0$, (b) $U_0 = 5$. The parameters are $q_0 = 1$, $q_1 = 0.1$, $R_{ON} = 1$, $R_{OFF} = 5$, $\sigma^2 = 1$, $\tau_c = 1$.

Corresponding to (15) the PDF of the memristance in this case reads

$$P_R(r,t) = \frac{1}{\sqrt{4\pi D(t)}} \sum_k \frac{1}{|R'(q_k(r))|} \exp\left\{-\frac{q_k^2(r)}{4D(t)}\right\}. \tag{16}$$

Here we used the same notations as in (6).

For considering case of an exponential dependence of the memristance on charge from (8) and (16) we obtain

$$P_R(r,t) = \frac{1}{\sqrt{4\pi D(t)}} \frac{q_0 \Delta R}{(R_{OFF} - r)(r - R_{ON})} \times \exp\left\{-\frac{1}{4D(t)}\left[q_1 + q_0 \ln\frac{R_{OFF} - r}{r - R_{ON}}\right]^2\right\}. \tag{17}$$

The 2D plot of PDF of the memristance for the case of white Gaussian fluctuations of the current is depicted in Fig. 3. As follows from Fig. 3, the initial unimodal probability distribution of the resistance becomes bimodal with increasing the time of observation. Two peaks correspond to low and high resistance states and manifest switching between them caused by fluctuations of the current.

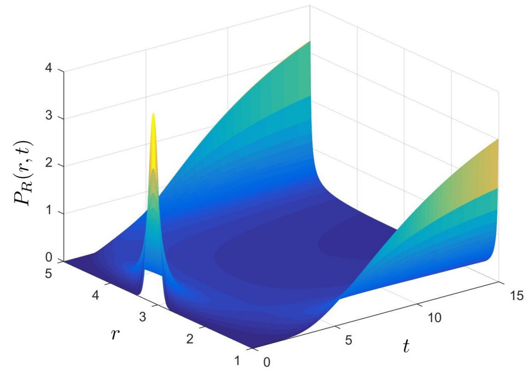


Fig. 3. Probability distribution of resistance (17) for the case of white Gaussian noise excitation $D(t) = Dt$ as a function of resistance $r$ and time $t$. The parameters are $q_0 = 1$, $q_1 = 0.1$, $R_{ON} = 1$, $R_{OFF} = 5$, $D = 0.5$.

## IV. Current-Controlled Memristor

Further we study a current-controlled ideal memristor. Based on the theoretical model recently described in [2], we analyze the same characteristics as for considered charge-controlled memristor. Specifically, we have the following system of equations

$$U(t) = (R_{ON} l(t) + R_{OFF}(1 - l(t)))\, I(t), \tag{18}$$

$$\frac{dl(t)}{dt} = \frac{\mu_V R_{ON}}{L^2} I(t), \tag{19}$$

where $l(t)$ is the normalized size of the doped region ($l(t) \in [0,1]$), $\mu_V$ is the average ion mobility, $L$ is the full size of the memristor with two states: low resistance $R_{ON}$ and high resistance $R_{OFF}$ ($R_{ON} \ll R_{OFF}$).

We study the case of the applied current $I(t)$ in the form of white Gaussian noise with non-zero mean $I_0$ and the intensity $2D_1$. The charge $q(t)$ is again Gaussian process, but according to (19), the probability distribution of the bounded random process $l(t)$ is non-Gaussian and contains two delta functions

$$P_l(y,t) = p_1(t)\delta(y) + p_2(t)\delta(y-1) + \frac{1}{\sqrt{4\pi D_1 c_0^2 t}} \exp\left\{-\frac{(y - c_0 I_0 t)^2}{4 D_1 c_0^2 t}\right\} 1_{(0,1)}(y), \tag{20}$$

where: $c_0 = \mu_V R_{ON}/L^2$, $1_A(y)$ is the indicator of set $A$ and

$$p_1(t) = \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi D_1 c_0^2 t}} \exp\left\{-\frac{(y - c_0 I_0 t)^2}{4 D_1 c_0^2 t}\right\} dy,$$

$$p_2(t) = \int_{1}^{\infty} \frac{1}{\sqrt{4\pi D_1 c_0^2 t}} \exp\left\{-\frac{(y - c_0 I_0 t)^2}{4 D_1 c_0^2 t}\right\} dy.$$

Based on the same technique we calculate from (18) and (20) the PDF of the resistance as

$$P_R(r,t) = \frac{1}{\Delta R} P_l\left(\frac{R_{OFF} - r}{\Delta R}, t\right), \tag{21}$$

where $\Delta R = R_{OFF} - R_{ON}$.

Analysis of time evolution of the PDF obtained allows us to understand the properties of the considered device.

## V. Conclusions

Two models of an ideal memristor with the external Gaussian noise have been under consideration. For both cases we have obtained exact analytical expressions for the PDF of the memristance. We have shown that for charge-controlled memristor the shape of the probability density function of resistance depends on what is applied in the form of Gaussian noise, the voltage or the current. Also, different noise excitations in the form of white and colored Gaussian noise have been analyzed. In the specific example of an exponential dependence of the resistance on the charge the influence of the noise mean value and the type of driven Gaussian noise on the memristor's switchings between two states has been found.


## Acknowledgment

This work was supported by the Government of the Russian Federation through Agreement No. 074-02-2018-330 (2).


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