Quotient method for stabilizing a ball-on-a-wheel system – Experimental results

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***Abstract*— This paper extends the quotient method proposed in [1] and applies it to stabilize a "ball-on-a-wheel" system. The quotient method requires a diffeomorphism to obtain the normal form of the input vector field and uses canonical projection to obtain the quotient. However, the whole process can be done without computing the normal form, which requires defining a quotient generating function and a quotient bracket. This paper presents the steps necessary to apply the quotient method without obtaining the normal form. Furthermore, a Lyapunov function is introduced to prove stability. This paper also presents the experimental implementation of the quotient method to stabilize a ball-on-a-wheel system.**

I. INTRODUCTION

There are many nonlinear systems used for pedagogical purposes. These systems serve not only as hands-on experiments for teaching purposes but also as test benches for new control laws. A good example is the inverted pendulum [2], [3], on which various nonlinear control methodologies such as approximate feedback linearization [4], adaptive backstepping [5], immersion and invariance [6], and quotient method [7] have been demonstrated. Other well-known pedagogical systems are the acrobot [8], [9], the ball-on-beam and the ball-on-a-wheel.

In this paper, the quotient method proposed in [1] is implemented and validated on a ball-on-a-wheel system. The setup consists of a wheel driven by an electric motor and a ball rolling in a groove on the periphery of the wheel. The objective is to keep the ball balanced on the wheel. This system is nonlinear, under-actuated, and open-loop unstable. The control strategies that have been demonstrated on this system include H_∞ PID control [10], flatness-based control [11] and full-state feedback linearization [12].

The particularity in this paper is that the the quotient method is implemented without resorting to a diffeomorphism. The algorithmic method includes two stages: in the forward stage, the quotients are defined by choosing 1-forms and are then used to obtain the required control law during the backward stage. The stability of the closed-loop system is proven by the existence of a Lyapunov function.

The paper is organized as follows. Section II develops the theory required by the algorithm. Section III briefly introduces the model used for control design, while Section IV designs the controller. A stability proof is given in Section V, while Section VI presents the experimental results. Finally, concluding remarks are given in Section VII.

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II. THEORICAL DEVELOPMENT

Consider a system defined on $\mathbb{R}^n$. The tangent bundle and the cotangent bundle are given by $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, respectively. Next,consider the exact $1-$form ω (a section of the cotangent bundle) and the vector field (a section of the tangent bundle) g such that $\omega(g)|_p \neq 0 \ \forall p \in \mathbb{R}^n$. Hereafter, for simplicity of notation, a 1-form acting on the vector field $\omega(g)$ will be written as ωg. Consider the subspace defined by the $1-$form ω as the collection of all vector fields p such that $\omega p = 0$. A projection can be define for the arbitrary vector field m on the subspace defined by ω along g as

Definition 1 (ω-projection): Consider the two arbitrary vector fields m and g and the 1-form ω such that $\omega g \neq 0$. Then, the projection of m along g on ω is given by

$$Pr_{\omega,g}(m) = m - \frac{\omega m}{\omega g} g.$$

Based on this definition, we can define an equivalence relationship as

Definition 2: The two vector fields m_1 and m_2 are equivalent ($m_1 \sim m_2$) for the given vector field g and the 1-form ω such that $\omega\, g \neq 0$ if and only if $Pr_{\omega,g}(m_1) = Pr_{\omega,g}(m_2)$.

It can be shown that the relation defined above is an equivalence relation as it satisfies the following properties:

- Reflexivity: $m_1 \sim m_1$. Since, $Pr_{\omega,g}(m_1) = Pr_{\omega,g}(m_1)$.
- Symmetry: if $m_1 \sim m_2$, then $m_2 \sim m_1$. Since, if $Pr_{\omega,g}(m_1) = Pr_{\omega,g}(m_2)$, then $Pr_{\omega,g}(m_2) = Pr_{\omega,g}(m_1)$.
- Transitivity: if $m_1 \sim m_2$ and $m_2 \sim m_3$, then $m_1 \sim m_3$. Since, if $Pr_{\omega,g}(m_1) = Pr_{\omega,g}(m_2)$ and $Pr_{\omega,g}(m_2) = Pr_{\omega,g}(m_3)$, then $Pr_{\omega,g}(m_1) = Pr_{\omega,g}(m_3)$.

Equivalence between the two vector fields m_1 and m_2 can also be identified if $m_1 - m_2 = \alpha\, g$, where α is a scalar function:

$$\begin{aligned} Pr_{\omega,g}(m+\alpha g) &= (m+\alpha g) - \frac{\omega(m+\alpha g)}{\omega g} g \\ &= m - \frac{\omega m}{\omega g} g + \alpha g - \frac{\alpha \omega g}{\omega g} g \\ &= m - \frac{\omega m}{\omega g} g + \alpha g - \alpha g \\ &= Pr_{\omega,g}(m). \end{aligned}$$

Hence, if $m_2 - m_1 = \alpha\, g$, then $Pr_{\omega,g}(m_2) = Pr_{\omega,g}(m_1 + \alpha\, g) = Pr_{\omega,g}(m_1)$, which implies $m_1 \sim m_2$.

The representative of the equivalence class is chosen to be the vector field that belongs to the subspace defined by ω. Hence, the quotient is the subspace defined by ω and the equivalence class is defined as:

$$[m_1] = \{m | m = m_1 + \alpha g \forall \alpha \text{ and } m_1 \omega = 0\}.$$

For the quotient method, it is required to compute the Lie bracket of the representatives from the members of the equivalence class. To this effect, the Müllhaupt bracket was defined in [13], with the definition repeated here.

Definition 3 (Müllhaupt (quotient) bracket): The Müllhaupt bracket of the two arbitrary vector fields m_1 and m_2, for the given input vector field g and the chosen integrable 1-form ω such that $\omega g \neq 0$, is given by:

$$< m_1, m_2 >_{\omega,g} = [m_1, m_2] + \frac{\omega m_2}{\omega g}[g, m_1] - \frac{\omega m_1}{\omega g}[g, m_2].$$

In the above definition, since ω appears both in the numerator and denominator, the 1-form ω can be assumed to be exact without any repercussion. Moreover, in the control design stage, exactness will play an important role. Next, the following theorem allows computing the Lie bracket of the representatives.

Theorem 1: Projection of the Müllhaupt bracket of two vector fields m_1 and m_2 gives the Lie bracket of the projection of individual vector fields, that is,

$$Pr_{\omega,g}(< m_1, m_2 >_{\omega,g}) = [Pr_{\omega,g}(m_1), Pr_{\omega,g}(m_2)]$$

Proof: Consider

$$\begin{aligned} & Pr_{\omega,g}(< m_1, m_2 >_{\omega,g}) \\ = \;& Pr_{\omega,g}([m_1, m_2] + \frac{\omega m_2}{\omega g}[g, m_1] - \frac{\omega m_1}{\omega g}[g, m_2]) \\ = \;& [m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2] \\ - \;& \frac{\omega([m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2])}{\omega g} g \qquad (1) \end{aligned}$$

where

$$\alpha_1 := \frac{\omega m_1}{\omega g}, \qquad \alpha_2 := \frac{\omega m_2}{\omega g}.$$

Also,

$$\begin{aligned} & \omega([m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2]) \\ = \;& \omega[m_1, m_2] + \alpha_2 \omega[g, m_1] - \alpha_1 \omega[g, m_2]. \end{aligned}$$

For the two arbitrary vector fields ζ_1 and ζ_2 and the exact 1-form ω, one has:

$$\omega[\zeta_1, \zeta_2] = L_{\zeta_1}\omega\zeta_2 - L_{\zeta_2}\omega\zeta_1.$$

Hence,

$$\begin{aligned} & \omega[m_1, m_2] + \alpha_2\omega[g, m_1] - \alpha_1\omega[g, m_2] \qquad (2) \\ = \;& L_{m_1}(\omega m_2) - L_{m_2}(\omega m_1) + \alpha_2 L_g(\omega m_1) \\ - \;& \alpha_2 L_{m_1}(\omega g) - \alpha_1 L_g(\omega m_2) + \alpha_1 L_{m_2}(\omega g). \qquad (3) \end{aligned}$$

Next, consider

$$\begin{aligned} & L_{m_1}(\omega m_2) - \alpha_2 L_{m_1}(\omega g) \\ = \;& L_{m_1}(\omega m_2) - \frac{\omega m_2}{\omega g} L_{m_1}(\omega g) \\ = \;& \omega g \frac{\omega g L_{m_1}(\omega m_2) - \omega m_2 L_{m_1} \omega g}{(\omega g)^2} \\ = \;& \omega g L_{m_1} \alpha_2, \qquad (4) \end{aligned}$$

as well as

$$L_{m_2}(\omega m_1) - \alpha_1 L_{m_2}(\omega g) = \omega g L_{m_2} \alpha_1, \qquad (5)$$

and

$$\begin{aligned} & \alpha_2 L_g(\omega m_1) - \alpha_1 L_g(\omega m_2) \\ = \;& \alpha_2 L_g(\omega m_1) - \alpha_2\alpha_1 L_g(\omega g) \\ - \;& \alpha_1 L_g(\omega m_2) + \alpha_2\alpha_1 L_g(\omega g) \\ = \;& \alpha_2(L_g(\omega m_1) - \alpha_1 L_g(\omega g)) \\ - \;& \alpha_1(L_g(\omega m_2) - \alpha_2 L_g(\omega g)) \\ = \;& \alpha_2(\omega g)L_g(\alpha_1) - \alpha_1(\omega g)L_g(\alpha_2). \qquad (6) \end{aligned}$$

Substituting (4), (5) and (6) into (3) modifies the LHS as

$$\begin{aligned} & Pr_{\omega,g}(< m_1, m_2 >_{\omega,g}) \\ = \;& [m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2] \\ - \;& \frac{\omega g L_{m_1}\alpha_2 - \omega g L_{m_2}\alpha_1}{\omega g} g \\ - \;& \frac{\alpha_2(\omega g)L_g(\alpha_1) - \alpha_1(\omega g)L_g(\alpha_2)}{\omega g} g \\ = \;& [m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2] \\ - \;& (L_{m_1}\alpha_2 - L_{m_2}\alpha_1 + \alpha_2 L_g\alpha_1 - \alpha_1 L_g\alpha_2)g. \qquad (7) \end{aligned}$$

Next, consider the RHS,

$$\begin{aligned} & [Pr_{\omega,g}(m_1), Pr_{\omega,g}(m_2)] \\ = \;& [m_1 - \alpha_1 g, m_2 - \alpha_2 g] \\ = \;& [m_1, m_2] - [m_1, \alpha_2 g] - [\alpha_1 g, m_2] + [\alpha_1 g, \alpha_2 g] \\ = \;& [m_1, m_2] - \alpha_2[m_1, g] - (L_{m_1}\alpha_2)g - \alpha_1[g, m_2] \\ + \;& (L_{m_2}\alpha_1)g + \alpha_1\alpha_2[g, g] + \alpha_1(L_g\alpha_2)g - \alpha_2(L_g\alpha_1)g \\ = \;& [m_1, m_2] + \alpha_2[g, m_1] - \alpha_1[g, m_2] \\ + \;& (-L_{m_1}\alpha_2 + L_{m_2}\alpha_1 + \alpha_1 L_g\alpha_2 - \alpha_2 L_g\alpha_1)g. \qquad (8) \end{aligned}$$

Comparing (7) and (8) shows that

$$Pr_{\omega,g}(< m_1, m_2 >_{\omega,g}) = [Pr_{\omega,g}(m_1), Pr_{\omega,g}(m_2)],$$

which proves the theorem. ■

Next, if $Pr_{\omega,g}$ is considered as a *push forward* operator for vector fields, then there exists a corresponding *pull back* operator for the $1-$form defined on the dual of the quotient obtained using the ω-projection. The following lemma defines such an *pullback* operator

Lemma 1: The pullback of the 1-form ω_N defined on the dual of the quotient generated using $Pr_{\omega,g}$ for the 1-form ω and the vector field g with $\omega g \neq 0$ is given by:

$$Pr^*_{\omega,g}(\omega_N) = \omega_N - \frac{\omega_N g}{\omega g}\omega. \qquad (9)$$

Proof: A 1-form acting on a vector field gives a scalar function invariant under mapping. Hence, for the arbitrary vector field m, $Pr^*_{\omega,g}$ must satisfy

$$Pr^*_{\omega,g}(\omega_N)m = \omega_N Pr_{\omega,g}(m).$$

Using (9), the LHS gives

$$\begin{aligned} Pr^*_{\omega,g}(\omega_N)m &= (\omega_N - \frac{\omega_N g}{\omega g}\omega)m \\ &= \omega_N m - \frac{\omega_N g}{\omega g}\omega m. \qquad (10) \end{aligned}$$

Next, substituting the definition of $Pr_{\omega,g}$ in the RHS gives

$$\begin{aligned} \omega_N Pr_{\omega,g} m &= \omega_N (m - \frac{\omega m}{\omega g} g) \\ &= \omega_N m - \frac{\omega m}{\omega g} \omega_N g. \end{aligned} \tag{11}$$

Comparing (10) and (11) shows that (9) represents the pullback of the 1-form defined on the quotient. ■

It is interesting to note that $Pr_{\omega,g}$ is an anchor and that the Müllhaupt bracket satisfies the axioms required to define a Lie Algebroid [14], [15]. However, due to space limitation, the proof is not included in this paper. The next section briefly introduces the model of the system.

III. Ball-on-a-wheel Model

The system consists of a ball running on a grooved wheel. The states of the system are the angular position of the ball with respect to the vertical axis θ_1, the corresponding velocity $\dot{\theta}_1$, the wheel position and the wheel velocity (θ_2 and $\dot{\theta}_2$), as shown in Figure 1. A model was derived

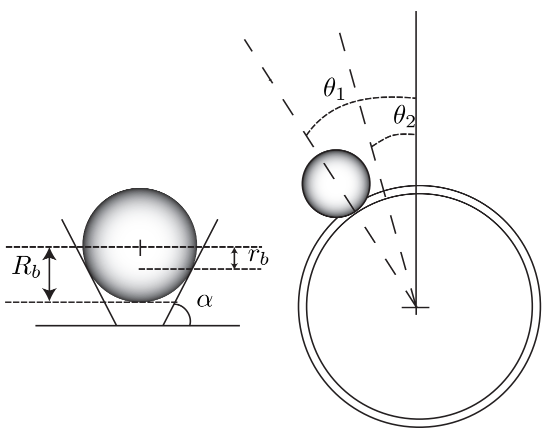

Fig. 1. Schematic representation of the ball-on-a-wheel system

using analytical mechanics. The results of [12] were slightly modified to include the influence of the groove angle (α as shown in Figure 1), leading to the following equation:

$$\dot{x} = \begin{pmatrix} x_2 \\ ax_4 + bsinx_1 \\ x_4 \\ px_4 + qsinx_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ 0 \\ r \end{pmatrix} u, \tag{12}$$

with the states defined as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{pmatrix}.$$

The model parameters are given in Table I, and the physical parameters in Table II.

a	$-i_m^2 \frac{2r_w K_m^2}{R_a(r_b+r_w)I_{tot}}$	p	$-i_m^2 \frac{K_m^2(5\cos(\alpha)^2+2)}{R_a I_{tot}}$
b	$\frac{g(5I_w+2r_w^2 m_b)}{(r_b+r_w)I_{tot}}$	q	$\frac{2gr_w m_b}{I_{tot}}$
c	$i_m \frac{2r_w K_m}{R_a(r_w+r_b)I_{tot}}$	r	$i_m \frac{K_m(5\cos(\alpha)^2+2)}{R_a I_{tot}}$
I_{tot}	$I_w(5cos(\alpha)^2+2)+2r_w^2 m_b$		

TABLE I
Model parameters

r_w	Wheel radius [m]
r_b	rolling ball radius ($r_b = cos(\alpha)R_b$) [m]
R_b	Ball radius [m]
α	Groove angle [rad]
m_b	Ball mass [kg]
I_w	Wheel inertia [kgm^2]
K_m	Motor torque constant [Nm/A]
R_m	Motor winding resistance [Ω]
i_m	Gearbox transmission ratio [-]
g	Gravitational acceleration [m/s^2]

TABLE II
Physical parameters

Note that the coefficients are not completely independent of each other, as for example:

$$ar = cp. \tag{13}$$

It can be shown that the system model is feedback linearisable [12] with one of the feedback linearising output being

$$h(\mathbf{x}) = rx_1 - cx_3,$$

in the domain

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^4 | -\frac{\pi}{2} < x_1 < \frac{\pi}{2} \right\}. \tag{14}$$

This domain is inherent to the system because beyond Ω there is no force to keep the ball sticking to the wheel. Hence, it is impossible to design a control law that could stabilize beyond Ω.

IV. Controller Design

The quotient method is an algorithmic method to design control laws for nonlinear systems [1]. It consists of two stages. The forward stage determines a diffeomorphism that reduces the effect of the control input on the last state. The last state is then removed from the system and considered as a new virtual control input to the remaining subsystem. Then the process is repeated until the dimension is reduced to 1.

The backward stage then starts with the simplest subsystem and designs a control law for the virtual control input, which is trivial for a scalar system. This control law is then extended to the next subsystem obtained in the forward stage. This step is repeated until a stabilising control law is obtained for the full system.

A. Differences from the quotient method with diffeomorphism

In this paper, the quotient method will be applied without resorting to diffeomorphisms. All previous applications of the quotient method [8], [7] required diffeomorphisms to achieve the normal form of the input vector field. The main characteristic of the quotient method with diffeomorphism is that dimension reduction is achieved at each iteration. This is possible because achieving the normal form of the input vector field and using the canonical projection automatically defines a natural quotient on the base manifold $\mathbb{R}^n$ with the integral of the input vector field as the equivalence class.

Evidently, this is not possible if diffeomorphism is not used. There is no dimension reduction since at every iteration the dimension remains n. However, computing the diffeomorphism requires integrating a system of n 1- forms at each iteration, which in itself could be a complicated task for some input vector fields. Taking dimension reduction in account, the diffeomorphism version requires integrating $(n(n+1)/2-1)$ 1-forms for an n-dimensional system. In case of the present algorithm, only n 1-forms are required to be integrated. Moreover, the 1-forms that need to be integrated are chosen by the user. Hence, this algorithm greatly reduces the effort required in obtaining the control law. The theoretical contribution of this paper is to demonstrate the methodology, whereby solving the system of 1-form is avoided using Müllhaupt bracket and Theorem 1.

Although the steps are demonstrated for the ball-on-a-wheel system, the algorithm is generic and can be applied to any feedback linearizable system.

B. Forward stage

We begin with the model of the system

$$f_1 := \begin{pmatrix} x_2 \\ ax_4 + b\sin x_1 \\ x_4 \\ px_4 + q\sin x_1 \end{pmatrix} \qquad g_1 := \begin{pmatrix} 0 \\ c \\ 0 \\ \frac{cp}{a} \end{pmatrix}$$

which gives $fg_1 := [f_1, g_1] = \begin{pmatrix} -c \\ -c\,p \\ -\frac{c\,p}{a} \\ -\frac{c\,p^2}{a} \end{pmatrix}$.

Iteration 1:

- The process begins with choosing the exact 1-form ω_1, which satisfies $\omega_1 g_1 \neq 0$. The chosen ω_1 must also satisfy Lemma 3 of [1] in order to have affine system on the quotient. This lemma states that the chosen 1-form must satisfy $L_{g_1}\omega_1 g_1 = \omega_1 g_1 \kappa_{1,2}$. $\kappa_{1,2}$ is given by the equation $[g_1, [f_1, g_1]] = \kappa_{1,1}g_1 + \kappa_{1,2}[f_1, g_1]$. Clearly, it requires the distribution $\Delta =$span $\{g_1, [f_1, g_1]\}$ to be involutive. Substituting for f_1 and g_1 results in $\kappa_{1,2} = 0$, which qualifies $\omega_1 = (0, 1, 0, 0)$. This implies the integral of ω_1 is $\gamma_1 = x_2$+constant. Since convergence to the origin is required, all the integrals must preserve the origin. This implies that the constant is 0 and $\gamma_1 = x_2$.
- The quotient $\mathcal{Q}_1$ is the subspace defined by ω_1, and the representative of f_1 on the quotient is

$$f_2 := Pr_{\omega_1,g_1}(f_1) = \begin{pmatrix} x_2 \\ 0 \\ x_4 \\ \frac{\sin(x_1)\,(a\,q - b\,p)}{a} \end{pmatrix}.$$

- Since Lemma 2 and Lemma 3 of [1] are satisfied by the choice of ω_1, Lemma 4 of [1] can be used to define the input vector field of the quotient system:

$$g_2 := \frac{-1}{\omega_1 g_1} Pr_{\omega_1,g_1}(fg_1) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{a} \\ 0 \end{pmatrix}.$$

- The quantities required for the next iteration are $[f_2, g_2]$ and $[g_2, [f_2, g_2]]$. To this effect, Theorem 1 will be used:

$$\begin{aligned} fg_2 &:= [f_2, g_2] = [Pr_{\omega_1,g_1}(f_1), -\frac{1}{\omega_1, g_1} Pr_{\omega_1,g_1}(fg_1)] \\ &= Pr_{\omega_1,g_1} < f_1, -\frac{1}{\omega_1, g_1} fg_1 >_{\omega_1,g_1} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\cos(x_1)\,(a\,q - b\,p)}{a} \end{pmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} &gfg_2 := [g_2, [f_2, g_2]] \\ &= Pr_{\omega_1,g_1} < \frac{-1}{\omega_1, g_1} fg_1, < f_1, \frac{-1}{\omega_1 g_1} fg_1 >_{\omega_1,g_1} >_{\omega_1,g_1} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sin(x_1)\,(a\,q - b\,p)}{a} \end{pmatrix}. \end{aligned}$$

Note that all the vector fields are in $\mathcal{Q}_1$ and further iterations are carried out in $\mathcal{Q}_1$.

Iteration 2:

- The equation $gfg_2 = \kappa_{2,1}g_2 + \kappa_{2,2}fg_2$ yields $\kappa_{2,2} = \tan(x_1)$. This restricts the choice of ω_2 such that $L_{g_2}(\omega_2 g_2) = \omega_2 g_2 \tan(x_1)$. Solving this equation yields $\omega_2 g_2 = \cos(x_1)$, which in turn implies that $\omega_2 = (\cos(x_1)\ \ 0\ \ 0\ \ 0)$. The pullback of ω_2 gives $\omega_{2,1} := Pr^*_{\omega_1,g_1}(\omega_2) = (\cos(x_1)\ \ 0\ \ 0\ \ 0)$. Now, $\omega_{2,1}$ is exact and the corresponding integral is $\gamma_2 = \sin(x_1)$. The constant of integration is zero to preserve the origin.
- For this iteration, the quotient $\mathcal{Q}_2 \subset \mathcal{Q}_1$ is the intersection of the subspace defined by ω_2 and the subspace defined by ω_1. The representative of f_2 is

$$f_3 = Pr_{\omega_2,g_2}(f_2) = \begin{pmatrix} 0 \\ 0 \\ x_4 - \frac{p\,x_2}{a} \\ \frac{\sin(x_1)\,(a\,q - b\,p)}{a} \end{pmatrix}.$$

- The new input vector field is
$$g_3 = \frac{-1}{\omega_2 g_2} Pr_{\omega_2,g_2} f g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q - \frac{b\,p}{a} \end{pmatrix}.$$
- For the next iteration, $[f_3, g_3]$ needs to be computed using Theorem 1:
$$\begin{aligned} fg_3 &:= [f_3, g_3] = [Pr_{\omega_2,g_2} f_2, \frac{-1}{\omega_2 g_2} Pr_{\omega_2,g_2} f g_2] \\ &= Pr_{\omega_2,g_2} < f_2, \frac{-1}{\omega_2 g_2} f g_2 >_{\omega_2,g_2} \\ &= Pr_{\omega_2,g_2}([f_2, \frac{-1}{\omega_2 g_2} f g_2] - \frac{\omega_2 f g_2}{(\omega_2 g_2)^2}[g_2, f_2] \\ &- \frac{\omega_2 f_2}{\omega_2 g_2}[g_2, \frac{-1}{\omega_2 g_2} f g_2]). \end{aligned}$$

The individual terms can be calculated as:
$$[f_2, \frac{-1}{\omega_2 g_2} f g_2] =$$
$$[Pr_{\omega_1,g_1} f_1, \frac{-1}{\omega_2 g_2} Pr_{\omega_1,g_1} < f_1, -\frac{1}{\omega_1, g_1} f g_1 >_{\omega_1,g_1}]$$
$$= Pr_{\omega_1,g_1} < f_1, \frac{-1}{\omega_2 g_2} < f_1, -\frac{1}{\omega_1, g_1} f g_1 >_{\omega_1,g_1} >_{\omega_1,g_1}$$
$$[g_2, f_2] = -f g_2$$
$$[g_2, \frac{-1}{\omega_2 g_2} f g_2] =$$
$$[\frac{-1}{\omega_1 g_1} Pr_{\omega_1,g_1} f g_1, \frac{-1}{\omega_2 g_2}$$
$$Pr_{\omega_1,g_1} < f_1, -\frac{1}{\omega_1, g_1} f g_1 >_{\omega_1,g_1}]$$
$$= Pr_{\omega_1,g_1} < \frac{-1}{\omega_1, g_1} f g_1, \frac{-1}{\omega_2 g_2}$$
$$< f_1, -\frac{1}{\omega_1, g_1} f g_1 >_{\omega_1,g_1} >_{\omega_1,g_1}.$$
Substituting these term gives
$$f g_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{b\,p}{a} - q \\ 0 \end{pmatrix}.$$

<u>Iteration 3:</u>

- Since computing $[g_3, [f_3, g_3]]$ through Müllhaupt bracketing is a tedious task, we will resort to another method to find $\kappa_{3,2}$. To this effect, if we can find the exact 1-form ω_{33} (i.e. $Pr^*_{\omega_1,g_1}(Pr^*_{\omega_2,g_2}(\omega_{33}))$ is exact) such that $\omega_{33} g_3 = 0$, then it follows:
$$\begin{aligned} & [g_3, [f_3, g_3]] = \kappa_{3,1} g_3 + \kappa_{3,2}[f_3, g_3] \\ & \Rightarrow \omega_{33}[g_3, [f_3, g_3] = \kappa_{3,1}\omega_{33} g_3 + \kappa_{3,2}\omega_{33}[f_3, g_3] \\ & \Rightarrow \omega_{33}[g_3, [f_3, g_3]] = \kappa_{3,2}\omega_{33}[f_3, g_3] \\ & \Rightarrow L_{g_3}\omega_{33}[f_3, g_3] - L_{[f_3,g_3]}\omega_{33} g_3 \\ & = \kappa_{3,2}(L_{f_3}\omega_{33} g_3 - L_{g_3}\omega_{33} f_3) \\ & \Rightarrow L_{g_3} L_{f_3}\omega_{33} g_3 - L^2_{g_3}\omega_{33} f_3 = -\kappa_{3,2} L_{g_3}\omega_{33} f_3 \\ & \Rightarrow L^2_{g_3}\omega_{33} f_3 = \kappa_{3,2} L_{g_3}\omega_{33} f_3. \end{aligned}$$

This equation is now used to determine $\kappa_{3,2}$. For simplicity of notation, we define:
$$\begin{aligned} \omega_{33,2} &:= Pr^*_{\omega_2,g_2}(\omega_{33}) \\ \omega_{33,1} &:= Pr^*_{\omega_1,g_1}(Pr^*_{\omega_2,g_2}(\omega_{33})). \end{aligned}$$
One such ω_{33} is $\omega_{33} = (0\ \ 0\ \ 1\ \ 0)$, since $\omega_{33} g_3 = 0$ and $\omega_{33,1} = (-p/a\ \ 0\ \ 1\ \ 0)$ is exact. For $Pr_{\omega_2,g_2}(Pr_{\omega_1,g_1}(g_3)) = g_3$, we can directly compute:
$$\begin{aligned} L_{g_3}(\omega_{33} f3) &= L_{g_3}(x_4 - \frac{p\,x_2}{a}) = q - \frac{bp}{a} \\ \Rightarrow L^2_{g_3}(\omega_{33} f_3) &= 0 \\ \Rightarrow \kappa_{3,2} &= 0 \end{aligned}$$
Computing $L_{g_3}\omega_{33} f_3$ in this manner is also possible since any element from the equivalence class
$$[g_3] = \{g_3 + \alpha_2 g_2 + \alpha_1 g_1, \forall \alpha_1 \text{ and } \alpha_2\},$$
satisfies
$$L_{[g_3]}\omega_{33} f_3 = L_(g_3 + \alpha_2 g_2 + \alpha_1 g_1)\omega_{33} f_3 = L_{g_3}\omega_{33} f_3,$$
for $L_{g_2}\omega_{33} f_3 = 0$ and $L_{g_1}\omega_{33} f_3 = 0$.

Lemma 2: For any exact 1-form ω_{33} such that $\omega_{33} g_3 = 0$, then
$$L_{g_2}\omega_{33} f_3 = 0$$
$$L_{g_1}\omega_{33} f_3 = 0$$
Proof: For any ω_{33} , it can be shown that
$$\begin{aligned} \omega_{33} g_2 - \frac{\omega_2 g_2}{\omega_2 g_2}\omega_{33} g_2 &= 0 \\ \Rightarrow \omega_{33} g_2 - \frac{\omega_{33} g_2}{\omega_2 g_2}\omega_2 g_2 &= 0 \\ \Rightarrow \left(\omega_{33} - \frac{\omega_{33} g_2}{\omega_2 g_2}\omega_2\right) g_2 &= 0 \\ \Rightarrow \omega_{33,2}\, g_2 &= 0. \end{aligned}$$
It follows from
$$\omega_{33}\, g_3 = 0$$
that
$$\omega_{33,2}[f_2, g_2] = 0$$
since $g_3 = -\frac{1}{\omega_2 g_2} Pr_{\omega_2, g2}[f_2, g_2]$
$$\begin{aligned} \Rightarrow L_{f_2}\omega_{33,2} g_2 - L_{g_2}\omega_{33,2} f_2 &= 0 \\ \Rightarrow L_{g_2}\omega_{33,2}\, f_2 = L_{g_2}\omega_{33}\, f_3 &= 0. \end{aligned}$$
Similarly, it can be shown that $\omega_{33,1} g_1 = 0$ and $\omega_{33,1}[f_1, g_1] = 0$. Using these two results, it is easy to show that $L_{g_1}\omega_{33} f_3 = 0$ ■

A suitable choice for ω_3 is $\omega_3 = (0\ \ 0\ \ 0\ \ 1)$, since it satisfies $L_{g_3}(\omega_3 g_3) = \kappa_{3,2}\omega_3 g_3$ and $Pr^*_{\omega_1,g_1}(Pr^*_{\omega_2,g_2}(\omega_3)) = (0\ -p/a\ 0\ 1)$ is exact. Hence, the integral is $\gamma_3 = \frac{-p}{a} x_2 + x_4$, which preserves the origin.

- $f_4 := Pr_{\omega_3,g_3} f_3 = \begin{pmatrix} 0 \\ 0 \\ x_4 - \frac{p\,x_2}{a} \\ 0 \end{pmatrix}$.
- $g_4 := \frac{-1}{\omega_3 g_3} Pr_{\omega_3,g_3} f g_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

It only remains to choose ω_4 such that $\omega_4 g_4 \neq 0$ and $Pr^*_{\omega_1,g_1}(Pr^*_{\omega_2,g_2}(Pr^*_{\omega_3,g_3}(\omega_4)))$ is exact. For $\omega_4 = (0\ 0\ 1\ 0)$, $Pr^*_{\omega_1,g_1}(Pr^*_{\omega_2,g_2}(Pr^*_{\omega_3,g_3}(\omega_4))) = (-p/a\ \ 0\ \ 1\ \ 0)$, which is exact and the corresponding integral is $\gamma_4 = \frac{-p}{a} x_1 + x_3$. Note that substituting (13) in γ_4 yields $-h(\mathbf{x})$, the feedback linearizing output, which is clearly seen in the algorithm with diffeomorphism [1].

C. Backward stage

In this stage, the quantities obtained during the forward stage will be used to design a stabilizing control law. The philosophy is the same as in [1]. Consider the diffeomorphism defined as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \gamma_4 \\ \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} x_3 - \frac{p\,x_1}{a} \\ x_4 - \frac{p\,x_2}{a} \\ \sin(x_1) \\ x_2 \end{pmatrix}.$$

This diffeomorphism converts the system into the strict feedback form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ (-((bp)/a) + q) z_3 \\ \sqrt{1 - z_3^2}\, z_4 \\ a z_2 + b z_3 + p z_4 + c u \end{pmatrix}. \qquad (15)$$

Since the backward stage designs a control law that will asymptotically stabilize (15) to the origin, it is required to preserve the correspondence between the origins of the $\mathbf{z}-$coordinates and the $\mathbf{x}-$coordinates while choosing the γ's. The idea behind the backward stage is to use the first equation to compute z_2 that stabilizes the first equation of (15) assuming z_2 as input. Then, we use iteratively the 2^{nd}, 3^{rd} and 4^{th} equations to determine stabilizing z_3, z_4 and u, respectively. For details, please refer to [1].

Iteration 1:

- Define $e_1 := \gamma_4 = x_3 - \frac{p\,x_1}{a}$.
- Compute the desired function for γ_3 as
$$\gamma_{3,d} = \frac{-k_1 e_1 - \omega_4 f_4}{\omega_4 g_4} + \gamma_3 = -k_1 \left(x_3 - \frac{p\,x_1}{a} \right),$$
where k_1 is a positive constant gain.
- Define $e_2 := \gamma_3 - \gamma_{3,d} = x_4 + k_1\,x_3 - \frac{p\,(x_2 + k_1\,x_1)}{a}$.
- Compute $\gamma_{2,d} = \frac{-k_2 e_2 + L_{f_1}(\gamma_{3,d}) - \omega_3 f_3}{\omega_3 g_3} + \gamma_2 = -\frac{a\,k_2\,x_4 + a\,k_1\,x_4 - k_2\,p\,x_2 - k_1\,p\,x_2 + a\,k_2\,k_1\,x_3 - k_2\,k_1\,p\,x_1}{a\,q - b\,p}$, where k_2 is a positive constant gain.
- Define $e_3 := \gamma_2 - \gamma_{2,d} = \sin(x_1) + \frac{a\,k_2\,x_4 + a\,k_1\,x_4 - k_2\,p\,x_2 - k_1\,p\,x_2 + a\,k_2\,k_1\,x_3 - k_2\,k_1\,p\,x_1}{a\,q - b\,p}$.
- Compute $\gamma_{1,d} = \frac{-k_3 e_3 + L_{f_1}(\gamma_{2,d}) - \omega_2 f_2}{\omega_2 g_2} + \gamma_1 = -\frac{\psi(\mathbf{x})}{a\,q\,\cos(x_1) - b\,p\,\cos(x_1)}$ where $\psi(\mathbf{x}) = a\,k_3\,k_2\,x_4 + a\,k_3\,k_1\,x_4 + a\,k_2\,k_1\,x_4 - k_3\,k_2\,p\,x_2 - k_3\,k_1\,p\,x_2 - k_2\,k_1\,p\,x_2 + a\,k_3\,q\,\sin(x_1) - b\,k_3\,p\,\sin(x_1) + a\,k_2\,q\,\sin(x_1) - b\,k_2\,p\,\sin(x_1) + a\,k_1\,q\,\sin(x_1) - b\,k_1\,p\,\sin(x_1) + a\,k_3\,k_2\,k_1\,x_3 - k_3\,k_2\,k_1\,p\,x_1$, and k_3 is a positive constant gain.
- Define $e_4 := \gamma_1 - \gamma_{1,d} = x_2 + \frac{\psi(\mathbf{x})}{a\,q\,\cos(x_1) - b\,p\,\cos(x_1)}$
- Compute $u = \frac{-k_4 e_4 + L_{f_1}(\gamma_{1,d}) - \omega_1 f_1}{\omega_1 g_1} = \mathcal{K}^T \mathcal{X}$, where
$$\mathcal{K} = \begin{bmatrix} 1 \\ k_1 + k_2 + k_3 \\ k_4 \\ k_1 k_2 + k_1 k_3 + k_2 k_3 \\ (k_1 + k_2 + k_3) k_4 \\ k_1 k_2 k_3 \\ (k_1 k_2 + k_1 k_3 + k_2 k_3) k_4 \\ k_1 k_2 k_3 k_4 \end{bmatrix}$$
and
$$\mathcal{X} = \begin{bmatrix} -\frac{a}{c} x_4 - \frac{b}{c} sin(x_1) \\ \frac{-x_2}{c cos(x_1)^2} \\ -\frac{x_2}{c} \\ -\frac{sin(x_1)(r x_2^2 - c x_4 x_2 + cos(x_1)(br - cq))}{c(br - cq) cos(x_1)^2} \\ \frac{-sin(x_1)}{c cos(x_1)} \\ -\frac{r x_2 - c x_4 - c x_2 x_3 tan(x_1) + r x_1 x_2 tan(x_1)}{c(br - cq) cos(x_1)} \\ \frac{c x_4 - r x_2}{c(br - cq) cos(x_1)} \\ \frac{c x_3 - r x_1}{c(br - cq) cos(x_1)} \end{bmatrix}, \qquad (16)$$
where k_4 is a positive constant gain.

This concludes the design of control law. The stability of the closed-loop system is discussed in the following section.

V. Discussion on Stability

It is much easier to establish stability if the closed-loop system is transformed using the diffeomorphism

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \Phi_e(\mathbf{x}) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

which uses the definitions of e_1 to e_4 obtained during the backward stage. The closed-loop system becomes:

$$\begin{pmatrix} \dot\epsilon_1 \\ \dot\epsilon_2 \\ \dot\epsilon_3 \\ \dot\epsilon_4 \end{pmatrix} = f_e = \begin{pmatrix} -k_1 \epsilon_1 + \epsilon_2 \\ -k_2 \epsilon_2 + C_2 \epsilon_3 \\ -k_3 \epsilon_3 + C_3 \epsilon_4 \\ -k_4 \epsilon_4 \end{pmatrix}, \qquad (17)$$

where $C_2 = \left(q - \frac{bp}{a} \right)$ and $C_3 = \cos(x_1)$, which in $\epsilon-$coordinates is $C_3 = \sqrt{1 - \frac{(a(\mathrm{k1}^2(-\epsilon 1) + \mathrm{k1}\epsilon 2 + \mathrm{k2}\epsilon 2 - q\epsilon 3) + bp\epsilon 3)^2}{(bp - aq)^2}}$. This implies that $|C_3| \leq 1$. Next, consider the Lyapunov function

$$V = \epsilon_1^2 + \frac{1}{k_1 k_2} \epsilon_2^2 + \frac{2 C_2^2}{k_1 k_2^2 k_3} \epsilon_3^2 + \frac{8 C_2^2}{k_1 k_2^2 k_3^2 k_4} \epsilon_4^2.$$

Computing the directional derivative along f_e gives

$$\dot{V} = -2k_1\epsilon_1^2 + 2\epsilon_1\epsilon_2 - \frac{2k_2}{k_1k_2}\epsilon_2^2 + \frac{2C_2}{k_1k_2}\epsilon_2\epsilon_3 - \frac{4C_2^2k_3}{k_1k_2^2k_3}\epsilon_3^2 + \frac{4C_2^2C_3}{k_1k_2^2k_3}\epsilon_3\epsilon_4 - \frac{16C_2^2k_4}{k_1k_2^2k_3^2k_4}\epsilon_4^2$$

$$\leq -2k_1\epsilon_1^2 + 2|\epsilon_1||\epsilon_2| - \frac{2}{k_1}\epsilon_2^2 + \frac{2|C_2|}{k_1k_2}|\epsilon_2||\epsilon_3| - \frac{4C_2^2k_3}{k_1k_2^2k_3}\epsilon_3^2 + \frac{4C_2^2|C_3|}{k_1k_2^2k_3}|\epsilon_3||\epsilon_4| - \frac{16C_2^2k_4}{k_1k_2^2k_3^2k_4}\epsilon_4^2.$$

Using Young's inequality for the cross terms, i.e. for any $p > 0$, $|a||b| \leq \frac{a^2}{2p} + \frac{pb^2}{2}$, yields:

$$2|\epsilon_1||\epsilon_2| \leq k_1\epsilon_1^2 + \frac{\epsilon_2^2}{k_1} \quad \text{here, } p = \frac{1}{k_1},$$

$$\frac{2|C_2|}{k_1k_2}|\epsilon_2||\epsilon_3| \leq \frac{|C_2|}{k_1k_2}\epsilon_2^2\left(\frac{k_2}{2|C_2|}\right) + \frac{|C_2|}{k_1k_2}\epsilon_3^2\left(\frac{2|C_2|}{k_2}\right) = \frac{1}{2k_1}\epsilon_2^2 + \frac{2C_2^2}{k_1k_2^2}\epsilon_3^2 \quad \text{here, } p = \frac{2|C_2|}{k_2},$$

$$\frac{4C_2^2|C_3|}{k_1k_2^2k_3}|\epsilon_3||\epsilon_4| \leq \frac{4C_2^2}{k_1k_2^2k_3}|\epsilon_3||\epsilon_4| \quad \text{since, } |C_3| \leq 1$$

$$\leq \frac{2C_2^2}{k_1k_2^2k_3}\epsilon_3^2\left(\frac{k_3}{4}\right) + \frac{2C_2^2}{k_1k_2^2k_3}\epsilon_4^2\left(\frac{4}{k_3}\right) = \frac{C_2^2}{2k_1k_2^2}\epsilon_3^2 + \frac{8C_2^2}{k_1k_2^2k_3^2}\epsilon_4^2 \text{ here, } p = \frac{4}{k_3}.$$

Substituting in $\dot{V}$ gives:

$$\dot{V} \leq -2k_1\epsilon_1^2 + k_1\epsilon_1^2 + \frac{\epsilon_2^2}{k_1} - \frac{2}{k_1}\epsilon_2^2 + \frac{1}{2k_1}\epsilon_2^2 + \frac{2C_2^2}{k_1k_2^2}\epsilon_3^2 - \frac{4C_2^2}{k_1k_2^2}\epsilon_3^2 + \frac{C_2^2}{2k_1k_2^2}\epsilon_3^2 + \frac{8C_2^2}{k_1k_2^2k_3^2}\epsilon_4^2 - \frac{16C_2^2k_4}{k_1k_2^2k_3^2k_4}\epsilon_4^2$$

$$= -k_1\epsilon_1^2 - \frac{1}{2k_1}\epsilon_2^2 - \frac{C_2^2}{k_1k_2^2}\epsilon_3^2 - \frac{8C_2^2k_4}{k_1k_2^2k_3^2k_4}\epsilon_4^2.$$

Hence, since $\dot{V}$ is negative definite, the error system (17) is globally asymptotically stable(GAS). However, the original closed-loop system is not GAS since the diffeomorphism Φ_e is not defined everywhere. The determinant of $\frac{\partial \Phi_e}{\partial \mathbf{x}}$ is $\cos(x_1)$, which indicates that Φ_e is only valid in Ω as defined in (14). Hence, the controller is expected to work only in Ω.

VI. Experiments

In order to test the control law experimentally, a ball-on-a-wheel setup was built. It consists of a wheel driven by a DC motor as shown in Figure 2. The main challenge was to find a sensor capable of measuring the position of the ball on the wheel. Existing systems described in the literature seem to use either a contactless distance sensor that measures the horizontal displacement of the ball or a potentiometer-like setup that uses resistive wires over which the ball rolls. The former only works for small angles (i.e. for linear control), while the latter leads to practical problems such as measurement noise.

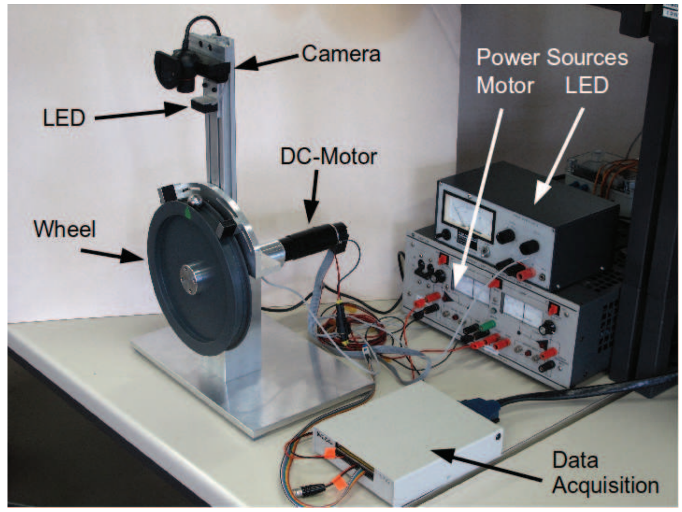

Fig. 2. Setup of the ball-on-a-wheel system

The solution chosen was to use computer vision. The main limitation is of course the sampling frequency. Normal webcams turned out to be too slow. We opted for the Sony PS3 Eye camera, which allows reaching up to 180 frames per second at a resolution of 320x240 pixels. The camera was mounted above the wheel (as shown in Figure 2) in order to see the movement of the ball on one single line of pixels, which simplified image processing considerably.

A. Results

The numerical values of the model parameters for the test setup are shown in the Table III.

a	b	c	p	q	r
-2.63	133.57	20.23	-4.75	4.94	36.49

TABLE III

Model parameters for the experimental setup

Quotient control led to the experimental results shown in Figure 3. The ball was initially placed at $45°$ angle. The controller was able to bring the ball to the equilibrium position. It also successfully rejected a perturbation introduced after 3.5s. Obtaining such relatively large angles was possible due to the high friction between the rubber ball running on a PVC wheel, which reduced slipping. Because, the camera was mounted on the top, there was a significant reduction in resolution for the angles measured beyond $45°$, which prevented stabilization when starting beyond that point. To be able to measure larger angles, efforts are being made to reduce the image processing time and to use the camera at different angle.

VII. Conclusions

This paper has extended the formulation of the quotient method to be used without resorting to normal forms of the input vector field. A projection function is introduced to define quotients and obtain the representative of a given vector field. Also, in order to compute the Lie bracket of the

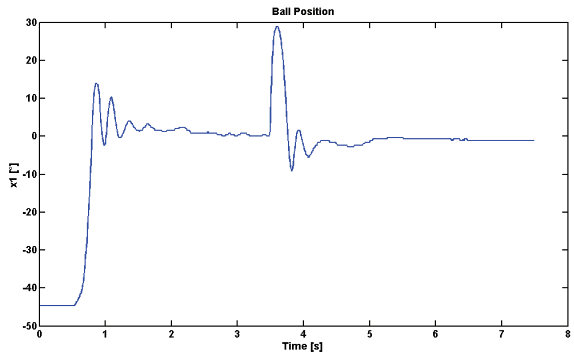

Fig. 3. Experimental swing-up and perturbation rejection with the quotient controller

representatives, the concept of Müllhaupt bracket is used. The development of the control law is done for a ball-on-a-wheel system. Furthermore, a Lyapunov function is proposed to prove the stability of the controlled system. These results are then verified experimentally. The setup that was built for this purpose includes a camera to detect the position of the ball. This turned out to be a reliable and inexpensive solution. The experimental region of attraction includes large ball angles, and stabilization from initial conditions of up to $45°$ was achieved.

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