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On the number of generators of the module of derivations and multiplicity of certain rings. (English summary)
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One attempt in classifying singularities is to study the associated local ring and the manner in which the singularity can be resolved using successive transformations (blow-ups) and rational maps from one higher-dimensional space to an intermediate space or to the space in which they initially lie. This can lead to various ways to codify the transformations derived, generally leading to graphs and trees. Another interesting attempt is to find a collection of objects that behaves sufficiently distinctively so as to distinguish between the type of singularities at hand, yet with enough generality so as to allow a computable criterion for classification. Such objects are for instance found in trying to compute invariants associated with the singularity. For surface singularities, a classification can be obtained after computing the ring of invariants associated with subgroups of $\mathrm{GL}_2(k)$.

The paper under review mainly conjectures a strong link between the module of derivations of the local ring structure and the multiplicity of this ring based on results that achieve a bound between the multiplicity and the order of the divisor class group of the complete local ring of the singularity. The conjecture is then checked on various low dimensional and intricate examples.

The interest in the module of derivations came from the observation that it plays a key role in the study of plane curves. Indeed, the exact number of generators for a plane curve can be found: Let $f\in k[[x,y]]$ be an irreducible power series of multiplicity ≥ 2, and let $R=k[[x,y]]/(f)$ be the corresponding factor ring; then the number of generators of the module of derivations $\mathrm{Der}\,R$ is exactly 2. This result has been established by D. P. Patil and B. Singh [Manuscripta Math. **68** (1990), no. 3, 327–335; MR1065934 (91g:14019)].

However, a similar result for the local ring $R=k[[X,Y,Z]]/(f)$ does not hold anymore. The authors then ask whether restricting the study to the ring of invariants of subgroups of $\mathrm{GL}_2(k)$ and trying to find the bound on the number of generators can shed light on this subject.

A key result obtained in the past (not involving derivations) is that of E. Brieskorn [Invent. Math. **4** (1967/1968), 336–358; MR0222084 (36 #5136)], who showed that the multiplicity of R, written $e(R)$, is always less than or equal to the order of the divisor class group, that is, $e(R)\leq |D_G(R)|$, when R is the ring of invariant of a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$. He obtained this result after completely classifying the subgroups of $\mathrm{GL}_2(\mathbb{C})$. He also gave the divisor class groups of the rings of invariants together with their multiplicities. Therefore, the key conjecture (Conjecture 2.2) of the paper seems natural, that is, for any rational surface singularity, $e(R)\leq |D_G(R)|$ always holds, where R is the corresponding complete local ring.

Illustrating the validity of this conjecture is undertaken after expressing the inequality using information on the module of derivations and on the fundamental cycle. To see this, there is a nice link between the divisor class group $D_G(R)$ and the intersection matrix $(C_i\cdot C_j)$ of a resolution of the singularity. Indeed, D. Mumford [Inst. Hautes Études Sci. Publ. Math. No. 9 (1961), 5–22; MR0153682 (27 #3643)] has shown that the topology of the space in which the singularity is embedded plays a key role in a classification attempt and established that $D_G(R)$ is isomorphic to $H_1(V-p,\mathbb{Z})$ and that $|D_G(R)| = |\det(C_i \cdot C_j)|$. The link between the multiplicity e of the rational singularity and the intersection matrix follows from a result of M. Artin [Amer. J. Math. **88** (1966), 129–136; MR0199191 (33 #7340)] that associates the fundamental cycle $\mathfrak{Z}$, obtained from the exceptional irreducible components of its resolution with the multiplicity, i.e. $-\mathfrak{Z}^2 = e$. The authors then prove that $-\mathfrak{Z}^2 \le |\det(C_i \cdot C_j)|$, from which follows many interesting particular applications such as for triple rational points, quotient singularities other than E_8, etc. For this last example it is also shown that $\mu(\operatorname{Der} R) \le 2|\frac{G}{[G,G]}| + 1$ using both Mumford's result $|\det(C_i \cdot C_j)| = |\frac{G}{[G,G]}|$ [op. cit.] and the bound $\mu(\operatorname{Der} R) \le 2e(\widehat{R}) + 1$, where $\widehat{R}$ is the completion of R with respect to its irrelevant maximal ideal, $R = k[X,Y]^G$ being the ring of invariants of a finite subgroup G of $\mathrm{GL}_2(k)$. The latter bound, involving the number of generators μ of the module of derivations $\operatorname{Der} R$, is obtained after combining a result obtained by E. Matlis [Proc. London Math. Soc. (3) **26** (1973), 273–288; MR0313247 (47 #1802)] (which states that when $(R,\mathfrak{m})$ is a pure 1-dimensional geometric local ring, then any ideal in R can be generated by $e(R)$ elements) with classical results on the theory of multiplicity that can be found in the book written by O. Zariski and P. Samuel [*Commutative algebra. Vol. II*, Reprint of the 1960 edition, Springer, New York, 1975; MR0389876 (52 #10706)].

The paper proves that Conjecture 2.2 holds for both what are called generalized quotient singularities and sandwiched singularities. These results are based on considerations about the dual intersection graph associated with a desingularization of rational surface singularities. Notice that rational surface singularities are the singularities of normal surfaces whose geometric genus does not change by desingularization. Starting from a quotient singularity of dimension 2, let $\mathcal{G}$ be the dual graph of the exceptional divisor and $\mathcal{R}$ be the dual graph obtained from $\mathcal{G}$ by reducing the self intersection of the irreducible components arbitrarily. $\mathcal{R}$ is called a generalized quotient singularity. The authors show that Conjecture 2.2 is true not only for generalized quotient singularities, but also for sandwiched singularity; namely a subgraph of a resolution of a smooth point. The proof rests partly on the statement that if the glueing of the rational trees $\mathcal{R}_1$ and $\mathcal{R}_2$ at E_1 and F_1 is rational then the coefficient of E_1 (resp. F_1) in the fundamental cycle of $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) is 1, which has been established by Lê Dũng Tráng and M. Tosun [Comment. Math. Helv. **79** (2004), no. 3, 582–604; MR2081727 (2005f:32046)].

The paper ends with various examples for which Conjecture 2.2 is satisfied, the conjecture being expressed using the number of generators of the module of derivations.

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References

1. M. Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966) 129–136. MR0199191 (33 #7340)
2. E. Brieskorn, Rationale singularitäten komplexer Flächen, Invent. Math. 4 (1968) 336–358. MR0222084 (36 #5136)

3. G.-M. Greuel, G. Pfister, H. Schönemann, Singular 3–0-20. A computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, http://www.singular.uni-kl.de/, March 2005.
4. R. Gurjar, M. Miyanishi, On the Jacobian conjecture for $\mathbf{Q}$-homology planes, J. Reine Angew. Math. 516 (1999) 115–132. MR1724617 (2001b:14094)
5. J. Herzog, M. Kühl, Maximal Cohen–Macaulay models over Gorenstein rings and Bourbaki sequences, in: Commutative Algebra and Combinatorics, Kyoto, 1985, in: Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 65–92. MR0951197 (89h:13029)
6. H. Krämer, Eine Bemerkung zu einer Vermutung von Lipman, Arch. Math. (Basel) 20 (1969) 30–35. MR0244237 (39 #5554)
7. E. Matlis, The multiplicity and reduction number of a one dimensional local ring, Proc. London Math. Soc. 26 (1973) 273–288. MR0313247 (47 #1802)
8. D. Mumford, The topology of normal singularities of an algebraic surface, Publ. Math. Inst. Hautes Études Sci. 9 (1961) 5–22. MR0153682 (27 #3643)
9. W. Neumann, Geometry of quasihomogeneous surface singularities, in: Singularities, Part 2, Arcata, Calif., 1981, in: Proc. Sympos. Pure Math., vol. 40, 1983, pp. 245–258. MR0713253 (85i:57008)
10. M. Neusel, L. Smith, Invariant Theory of Finite Groups, Math. Surveys Monogr., vol. 94, American Mathematical Society, 2002. MR1869812 (2002k:13012)
11. T. Okuma, Universal abelian covers of rational surface singularities, J. London Math. Soc. (2) 70 (2) (2004) 307–324. MR2078895 (2005e:14006)
12. D.P. Patil, Generators for the derivation module and the defining ideals of certain affine curves, PhD thesis, Tata Institute of Fundamental Research, Mumbai, 1989.
13. D.P. Patil, B. Singh, Generators for the derivation module and the relation ideals of certain curves, Manuscripta Math. 68 (1990) 327–335. MR1065934 (91g:14019)
14. O. Riemenschneider, Die Invariaten der endlichen Untergruppen, Math. Z. 153 (1977) 37–50. MR0447230 (56 #5545)
15. O. Riemenschneider, Zweidimensionale Quotientensingularitäten Gleichungen und Syzygien, Arch. Math. 37 (1981) 406–417. MR0643282 (83e:14002)
16. L.D. Tráng, M. Tosun, Combinatorics of rational singularities, Comment. Math. Helv. 79 (3) (2004) 582–604. MR2081727 (2005f:32046)
17. V. Wagh, On the modules of derivations, the modules of differentials and the multiplicity of certain rings, PhD thesis, Bhaskaracharya Pratishthana, Pune, 2006.
18. J. Watanabe, A note on Gorenstein rings of embedding codimension three, Nagoya Math. J. 50 (1973) 227–232. MR0319985 (47 #8526)
19. K.-i. Watanabe, Some remarks concerning Demazur's construction of normal graded rings, Nagoya Math. J. 83 (1981) 203–211. MR0632654 (83g:13016)
20. O. Zariski, P. Samuel, Commutative Algebra, vol. I, Grad. Texts in Math., vol. 28, Springer-Verlag, New York, 1975, reprint of the 1960 edition. MR0389876 (52 #10706)
21. O. Zariski, P. Samuel, Commutative Algebra, vol. II, Grad. Texts in Math., vol. 29, Springer-Verlag, New York, 1975, reprint of the 1960 edition. MR0389876 (52 #10706)