Fixed-order H_∞ Controller Design for Nonparametric Models by Convex Optimization [1]

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Abstract

A new approach for robust fixed-order H_∞ controller design by convex optimization is proposed. Linear time-invariant single-input single-output systems represented by nonparametric models in the frequency domain are considered. It is shown that the H_∞ robust performance condition can be represented by a set of linear or convex constraints with respect to the parameters of a linearly parameterized controller in the Nyquist diagram. Multimodel and frequency-domain uncertainty can be considered straightforwardly in the proposed approach. The proposed method is compared with the standard H_∞ control problem by a simulation example on an unstable system.

1 Introduction

Most controller design methods are based on plant parametric models. A parametric model can be obtained either by first principle modeling or by parameter estimation techniques using measured data. However, it is usually too difficult or time consuming to obtain a parametric model based on physical laws. On the other hand, identification of parametric models is based on several a priori information and user choices like sampling period, time-delay, number of parameters in numerator and denominator of plant and noise model, optimal excitation etc. For these reasons, some data-based methods (in time-domain or in frequency-domain) for controller design have been developed. Iterative Feedback Tuning (IFT) [1], Virtual Reference Feedback Tuning (VRFT) [2] and Iterative Correlation-based Controller Tuning (ICbT) [3] use time-domain data to tune fixed-order controllers for model reference and tracking problem. LQG control and optimal tracking in data space are considered in [4,5], respectively. A method for stability analysis of linear discrete time systems using only time-domain data is proposed in [6].

Frequency-domain data or spectral models can be easily obtained from input/output data using Fourier or spectral analysis [7]. These models are represented as a function of frequency ω and give some important information about the bandwidth and the static gain of the system. In this type of models the information is not condensed into a small set of parameters thus avoiding errors of unmodeled dynamics that appear in parametric models. Moreover, an estimate of the uncertainty due to noise can be readily computed.

Although spectral models are largely used in practice, controller design methods based on this type of models are rather limited. The first systematic controller design methods were based on loop shaping with graphical tools in Bode diagrams or in Nichols chart and are discussed in classical textbooks for design and analysis of control systems. The well-known Ziegler-Nichols tuning method based on only one point on the frequency response of the plant model (critical frequency) is still used to tune PID controllers in many practical situations. These approaches are very intuitive and work well for simple systems that can be approximated by a low-order model with relatively small delay. For unstable and nonminimum-phase systems and systems with parametric and frequency-domain uncertainty, more advanced methods should be used. Recently, it has been shown that the set of all stabilizing PID controllers achieving a desired gain and phase margin or H_∞ norm can be obtained using only the frequency-domain data [8]. Another frequency-domain method is the well-known Quantitative Feedback Theory (QFT) [9] which is based on loop shaping in the Nichols chart. Frequency-domain approaches lead usually to low-order controllers and the design procedures need some expertise and are based on trial and error. Although recently optimization ap-

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proaches are used to compute controllers in the QFT framework [10,11,12], H_2 and H_∞ control criteria for spectral models have not been considered.

With new progress in numerical methods for solving convex optimization problems, new approaches for controller design with convex objectives and constraints have been developed. These techniques have been also applied to controller design for spectral models. In [13,14] a convex optimization method for PID controller tuning by open-loop shaping in the frequency-domain is proposed. The infinity-norm of the difference between the desired open-loop transfer function and the achieved one weighted by a so-called target sensitivity function is minimized. For open-loop stable systems, it is shown through the small gain theorem that if the infinity norm is less than 1, then the nominal closed-loop system is stable. This is a sufficient condition which depends on the choice of the target sensitivity function. The condition for the stability of multiple models becomes more conservative as for each model a reasonable target sensitivity function should be available.

In [15] a robust fixed-order controller design using linear programming is proposed. The main feature of this method is that the stability and some robustness margins are guaranteed by linear constraints in the Nyquist diagram and the method is applicable to multiple models as well. However, the performance specifications are limited to the choice of a lower bound for crossover frequency and minimization of the integral of the tracking error. The results are improved by open-loop and closed-loop shaping using quadratic programming in [16].

In this paper, a new approach for robust fixed-order controller design is developed. It is shown that robust fixed-order linearly parameterized controllers for Linear Time Invariant Single-Input Single Output (LTI-SISO) systems represented by nonparametric spectral models can be computed by convex optimization. The performance specification, like the standard H_∞ control problem, is a constraint on the infinity norm of the weighted sensitivity function. It should be mentioned that the set of all fixed-order stabilizing controllers is a nonconvex set. In this paper, an inner convex approximation of this set is given by a set of linear constraints in the Nyquist diagram. The proposed method can be used for PID controllers as well as for higher order linearly parametrized controllers in discrete or continuous time. The case of unstable open-loop systems can also be considered if a stabilizing controller is available or the number of unstable poles of the plant is known. The main idea is to define new constraints such that the designed open-loop system has the winding number satisfying the Nyquist stability criterion. Another important feature is that, by contrast with the standard H_∞ problem, this approach can treat the case of multimodel uncertainty. The effectiveness of the proposed approach is illustrated by comparison with the standard H_∞ control design in a simulation example.

This paper is organized as follows: In Section 2 the class of models, controllers and the control objectives are defined. Section 3 introduces the control design methodology based on the linear and convex constraints in the Nyquist diagram. Simulation results and comparison with the standard H_∞ design are given in Section 4. Advantages and disadvantages of the proposed method are discussed in Section 5.

2 Problem Formulation

2.1 Class of models

The class of causal continuous-time LTI-SISO systems with bounded infinity norm is considered. It is assumed that the plant model belongs to a set $\mathcal{G}$ that contains m spectral models with multiplicative unstructured uncertainty :

$$\mathcal{G} = \left\{ G_i(j\omega)[1 + W_{2i}(j\omega)\Delta] \,;\, i = 1, \dots, m \,;\, \omega \in \mathbb{R} \right\} \tag{1}$$

where $W_{2i}(j\omega)$ is the uncertainty weighting frequency function and $\|\Delta\|_\infty < 1$. This type of models can be obtained from a parametric model or by spectral analysis from a set of input/output data.

Consider the input $u(t)$ and the output $y(t)$ of a discrete-time system $G(q^{-1})$ are available for a finite number of $t = 1, \dots, N$, where q^{-1} is backward shift operator. Assume that the data are noise-free and the initial and final conditions for u and y are zero, i.e $u(t) = y(t) = 0$ for $t \le 0$ and $t > N$. Then

$$G(e^{-j\omega}) = \frac{Y(\omega)}{U(\omega)} \tag{2}$$

where $U(\omega)$ and $Y(\omega)$ are the periodograms of $u(t)$ and $y(t)$ defined by [17]:

$$U(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-j\omega t}$$

$$Y(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} y(t) e^{-j\omega t}$$

For noisy data (2) gives the so-called Empirical Transfer Function Estimate (ETFE) which is asymptotically unbiased and has a variance of $\Phi_v(\omega)/|U(\omega)|^2$, with $\Phi_v(\omega)$ the spectrum of a stationary stochastic disturbance $v(t)$ at the output of the plant. In this case, the spectral model can be represented by a multiplicative uncertainty model $G(e^{-j\omega})[1 + W_2(e^{-j\omega})\Delta]$, where $|W_2(e^{-j\omega})|$ can be computed for a given probability level. It is clear that the quality of the ETFE estimate can be improved by different way of smoothing which are not discussed here.

In the sequel, for the sake of simplicity, we consider one of the models in $\mathcal{G}$ with multiplicative frequency-domain uncertainty, $G(j\omega)[1+W_2(j\omega)\Delta]$ and a continuous-time controller will be designed. Then the results are extended to the multimodel case and the convex combination of m spectral models. The results are also applicable to discrete-time models and other type of frequency-domain uncertainty.

2.2 Class of controllers

Linearly parameterized controllers are given by :

$$K(s,\rho) = \rho^T \phi(s) \tag{3}$$

where

$$\rho^T = [\rho_1, \rho_2, \ldots, \rho_n],$$
$$\phi^T(s) = [\phi_0(s), \phi_1(s), \ldots, \phi_{n-1}(s)],$$

n is the number of controller parameters and $\phi_i(s)$ are stable transfer functions with bounded infinity norm that may be chosen from a set of generalized orthonormal basis functions. Consider for example the Laguerre basis [18,19]:

$$\phi_0(s) = 1\,,\ \phi_i(s) = \frac{\sqrt{2\xi}(s-\xi)^{i-1}}{(s+\xi)^i} \quad \text{for } i \geq 1 \tag{4}$$

with $\xi > 0$. It can be shown that for any stable rational finite order transfer function $F(s)$ and for arbitrary $\varepsilon > 0$ there exists a sufficiently large n such that

$$\|F - \rho^T\phi\|_p < \varepsilon \quad \text{for} \quad 0 < p < \infty$$

Therefore, with this controller parameterization any finite order stable transfer function can be approximated with a desired accuracy by increasing the number of controller parameters. The quality of this approximation for a finite n, however, depends on the difference between the poles of $F(s)$ and ξ. An appropriate choice of ξ can lead to a better approximation for a given controller order. The optimal choice of basis functions has already been investigated in the context of modeling and identification [20] and will not be considered in this contribution. However, a practical guideline for an appropriate choice of the basis functions for an n-th order controller (with $n+1$ parameters) follows:

(1) Choose Laguerre basis with a large dimension, $n_{\max}+1$, and an arbitrary value for $\xi > 0$.
(2) Compute a $n_{\max}$-th order controller by the proposed optimization problem.
(3) Determine n dominant poles, $\xi_1, \ldots, \xi_n$, of the $n_{\max}$-order controller (using the model reduction techniques).
(4) Use the following generalized orthonormal basis functions for $i = 1, \ldots, n$ [19] :

$$\phi_0(s) = 1\,,\ \phi_i(s) = \frac{\sqrt{2R_e\{\xi_i\}}}{s+\xi_i} \prod_{k=1}^{i-1} \frac{s-\bar{\xi}_k}{s+\xi_k}$$

where $\bar{\xi}_k$ is the complex conjugate of ξ_k.

The main reason to use a linearly parameterized controller in this paper is that every point on the Nyquist diagram of the open-loop transfer function $L(j\omega,\rho)$ can be written as a linear function of the controller parameters ρ :

$$L(j\omega,\rho) = K(j\omega,\rho)G(j\omega) = \rho^T\phi(j\omega)G(j\omega) \tag{5}$$

This property helps obtaining a convex parameterization of fixed-order H_∞ controllers.

Remark: The bounded infinity-norm condition will be relaxed to allow the possible poles on the imaginary axis for plant model and controller. It is clear that, in this case, PID controllers belong to the set of parameterized controllers.

2.3 Design Specifications

Let the sensitivity function $\mathcal{S}(s) = [1+L(s)]^{-1}$ and the complementary sensitivity function $\mathcal{T}(s) = L(s)[1+L(s)]^{-1}$ be defined. The proposed approach can consider very simple specifications for the design of simple PID controllers as well as standard performance specifications for H_∞ control problems. For simple controller design, a lower bound on the modulus margin (the inverse of the infinity norm of the sensitivity function) and a desired value for the crossover frequency can be considered. Fixed-order controller can also be designed satisfying some constraints on the infinity norm of the weighted sensitivity functions.

A very standard robust control problem is to design a controller that satisfies $\|W_1\mathcal{S}\|_\infty < 1$ for a set of models, where $W_1(s)$ is the performance weighting filter. If the set of models is represented by multiplicative uncertainty, the necessary and sufficient condition for robust performance is given by [21]:

$$\big\||W_1\mathcal{S}| + |W_2\mathcal{T}|\big\|_\infty < 1 \tag{6}$$

There is no analytical solution to this problem, however, in the standard H_∞ framework a solution to the following approximate problem can be found:

$$\left\| \begin{matrix} W_1\mathcal{S} \\ W_2\mathcal{T} \end{matrix} \right\|_\infty < \frac{1}{\sqrt{2}} \tag{7}$$

This solution is conservative and leads to high order controllers. Moreover, it cannot be applied to systems with multimodel uncertainty.

The proposed approach in this paper is based on an infinite number of linear or convex constraints on the Nyquist diagram such that the following robust performance constraint is satisfied.

$$|W_1(j\omega)\mathcal{S}(j\omega)| + |W_2(j\omega)\mathcal{T}(j\omega)| < 1 \quad \forall\omega \qquad (8)$$

3 Robust Controller Design in Nyquist Diagram

3.1 Robust performance constraints

The basic idea is to represent the robust performance constraints in (8) in the Nyquist diagram and give a set of linear or convex constraints which guarantee that the robust performance condition is satisfied. This way, the controller design is represented by a convex feasibility problem.

Multiplying the robust performance condition in (8) by $|1 + L(j\omega, \rho)|$ gives:

$$|W_1(j\omega)| + |W_2(j\omega)L(j\omega,\rho)| < |1 + L(j\omega,\rho)| \quad \forall\omega \qquad (9)$$

Note that $|1+L(j\omega,\rho)|$ is the distance between the critical point and $L(j\omega,\rho)$. Hence, this constraint is satisfied if and only if there is no intersection in the Nyquist diagram between a circle centered at the critical point with a radius of $|W_1(j\omega)|$ and a circle centered at $L(j\omega,\rho)$ with a radius of $|W_2(j\omega)L(j\omega,\rho)|$ for all ω [21].

Now, consider a straight line $d^*(\omega)$ which is tangent to the circle with radius $|W_1(j\omega)|$ and orthogonal to the line between the critical point and $L(j\omega,\rho)$. Therefore, the robust performance condition in (8) is satisfied if and only if the circle centered at $L(j\omega,\rho)$ does not intersect $d^*(\omega)$ and is completely in the side that excludes the critical point (at the right hand side in Fig. 1). This condition cannot be represented as a convex constraint because $d^*(\omega)$ is a function of the controller parameters. Suppose that the frequency response of a desired open-loop transfer function, $L_d(j\omega)$, is available. Then, $d^*(\omega)$ can be approximated by $d(\omega)$ which is tangent to the circle with radius $|W_1(j\omega)|$ but orthogonal to the line connecting the critical point to $L_d(j\omega)$ (see Fig. 1). This will be a good approximation if $L_d(j\omega)$ is "close" to $L(j\omega,\rho)$.

It should be noted that the equation of $d(\omega)$ at each frequency depends only on $W_1(j\omega)$ and $L_d(j\omega)$. If we name x and y, respectively, the real and imaginary parts

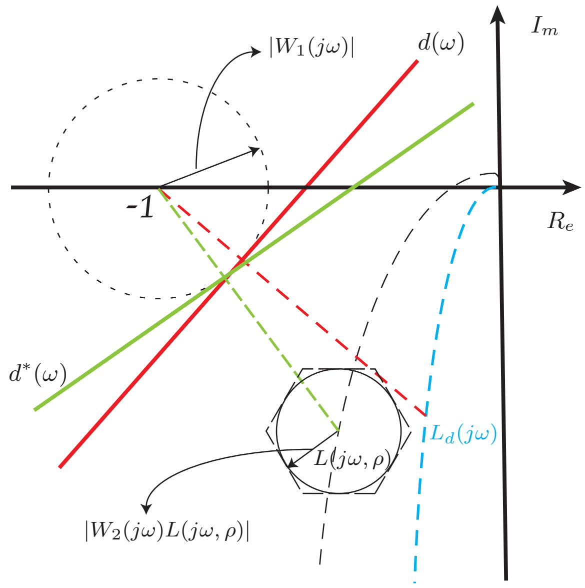

Figure 1. Linear constraints for robust performance in Nyquist diagram

of a point on the complex plane, the equation of $d(\omega)$ at each frequency becomes :

$$|W_1(j\omega)[1 + L_d(j\omega)]| - I_m\{L_d(j\omega)\}y - [1 + R_e\{L_d(j\omega)\}][1 + x] = 0 \qquad (10)$$

where $R_e\{\cdot\}$ and $I_m\{\cdot\}$ represent real and imaginary parts of a complex value, respectively. Therefore, the condition that $L(j\omega,\rho)$ for all ω is located in the side of $d(\omega)$ that excludes the critical point can be given by the following linear constraints :

$$|W_1(j\omega)[1 + L_d(j\omega)]| - I_m\{L_d(j\omega)\}I_m\{L(j\omega,\rho)\} - [1 + R_e\{L_d(j\omega)\}][1 + R_e\{L(j\omega,\rho)\}] < 0 \quad \forall\omega$$

Replacing $R_e\{L_d(j\omega)\} = 1/2[L_d(j\omega) + L_d(-j\omega)]$ and a similar expression for the imaginary part, the above linear constraints can be further simplified to:

$$|W_1(j\omega)[1 + L_d(j\omega)]| - R_e\{[1 + L_d(-j\omega)][1 + L(j\omega,\rho)]\} < 0 \quad \forall\omega \qquad (11)$$

There exists two alternatives in order that this condition to be satisfied for all models in the uncertainty set represented by a circle centered at $L(j\omega,\rho)$. The first alternative is to approximate the uncertainty circle by a polygon of $q > 2$ vertices. Then, the robust performance condition in (8) is satisfied if all vertices are located in the right side of $d(\omega)$. This can be represented by the following linear constraints :

$$
\begin{aligned}
&|W_1(j\omega)[1+L_d(j\omega)]| - \\
&\quad R_e\{[1+L_d(-j\omega)][1+L_i(j\omega,\rho)]\} < 0 \\
&\qquad\qquad \forall\omega \quad \text{and} \quad i = 1,\dots,q
\end{aligned} \tag{12}
$$

where $L_i(j\omega,\rho) = K(j\omega,\rho)G_i(j\omega)$ and

$$
G_i(j\omega) = G(j\omega)\left[1 + \frac{|W_2(j\omega)|}{\cos(\pi/q)} e^{j2\pi i/q}\right] \tag{13}
$$

It can be observed that the number of linear constraints are multiplied by q when the uncertainty circle is approximated by a polygon of q vertices.

The second alternative is to increase the radius of the performance circle by $|W_2(j\omega)L(j\omega,\rho)|$ which leads to the following convex constraints:

$$
\begin{aligned}
&|W_1(j\omega)[1+L_d(j\omega)]| + |W_2(j\omega)L(j\omega,\rho)[1+L_d(j\omega)]| - \\
&\quad R_e\{[1+L_d(-j\omega)][1+L(j\omega,\rho)]\} < 0 \quad \forall\omega
\end{aligned} \tag{14}
$$

This alternative has less constraints and no conservatism but leads to a bit more complex convex optimization problem (convex constraints instead of linear constraints).

The nonconvex constraint in (6) is convexified using a desired open-loop transfer function $L_d(s)$. In other words, the convex set in (14) is an inner approximation of the nonconvex set defined by the constraint in (6). The following Proposition shows under which condition a feasible point of the nonconvex set in (6) is also a feasible point of the convex inner approximation set in (14).

Proposition 1 *Consider that ρ° belongs to the nonconvex set (6), i.e. :*

$$
\| |W_1\mathcal{S}(\rho^\circ)| + |W_2\mathcal{T}(\rho^\circ)| \|_\infty = \gamma(\rho^\circ) < 1 \tag{15}
$$

then ρ° satisfies the constraints in (14) if and only if :

$$
\begin{aligned}
&|\angle(1+L_d(j\omega)) - \angle(1+L(j\omega,\rho^\circ))| < \\
&\cos^{-1}\left(|W_1(j\omega)\mathcal{S}(j\omega,\rho^\circ)| + |W_2(j\omega)\mathcal{T}(j\omega,\rho^\circ|)|\right) \quad \forall\omega
\end{aligned} \tag{16}
$$

The above inequality is satisfied if

$$
|\angle[1+L(j\omega,\rho^\circ)] - \angle[1+L_d(j\omega)]| < \cos^{-1}\gamma(\rho^\circ) \quad \forall\omega \tag{17}
$$

Proof: The proof is straightforward using the following relation:

$$
\begin{aligned}
&R_e\{[1+L_d(-j\omega)][1+L(j\omega,\rho^\circ)]\} = \\
&\qquad |1+L_d(-j\omega)||1+L(j\omega,\rho^\circ)|\cos\alpha
\end{aligned} \tag{18}
$$

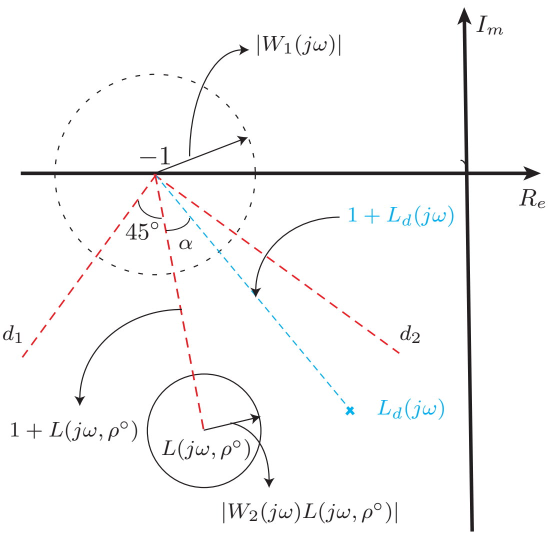

Figure 2. A graphical illustration of Proposition 1 for $\gamma(\rho^\circ) = 0.7$

where

$$
\alpha = |\angle[1+L(j\omega,\rho^\circ)] - \angle[1+L_d(j\omega)]| \tag{19}
$$

Replacing the right hand side of (18) in (14) gives:

$$
\begin{aligned}
&|W_1(j\omega)| + |W_2(j\omega)L(j\omega,\rho^\circ)| < \\
&\qquad |1+L(j\omega,\rho^\circ)|\cos\alpha \quad \forall\omega
\end{aligned} \tag{20}
$$

Dividing the both sides by $|1+L(j\omega,\rho^\circ)|$ leads to :

$$
|W_1(j\omega)\mathcal{S}(j\omega,\rho^\circ)| + |W_2(j\omega)\mathcal{T}(j\omega,\rho^\circ|)| < \cos\alpha \quad \forall\omega
$$

which is equivalent to (16). A sufficient condition for the above inequality is that the maximum value of the left hand side be smaller than $\cos\alpha$ or:

$$
\gamma(\rho^\circ) < \cos\alpha
$$

from which (17) can be concluded. ■

Suppose for example that ρ° is a feasible point of the nonconvex set with $\gamma(\rho^\circ) = 0.7$, then α, the phase difference of $1+L_d(j\omega)$ and $1+L(j\omega,\rho^\circ)$, should be less than $\cos^{-1}0.7 = 45°$. This represents a very large set (one quarter of the complex plane) of admissible $L_d(j\omega)$ for which ρ° is in the feasibility set of the inner approximation (see Fig. 2). It is clear that if the specifications are too tight so that for any feasible point ρ°, $\gamma(\rho^\circ)$ is very close to 1, the non convex set in (6) is too small and finding an inner approximation by the choice of L_d becomes very difficult. However, milder specifications leads to a larger nonconvex set in (6) and a reasonable choice of L_d leads usually to a nonempty inner approximation of the nonconvex set.

3.2 Main result

The main result of this section is presented in the following theorem:

Theorem 1 *Given the set of models $\mathcal{G}$ in (1) with performance weighting functions $W_{1i}(j\omega)$, the linearly parameterized controller in (3) stabilizes all models in $\mathcal{G}$ and satisfies the following robust performance condition:*

$$\| |W_{1i}\mathcal{S}_i| + |W_{2i}\mathcal{T}_i| \|_\infty < 1 \quad \textit{for} \quad i = 1, \dots, m \tag{21}$$

if

$$|W_{1i}(j\omega)| + |W_{2i}(j\omega)\rho^T\phi(j\omega)G_i(j\omega)| - \frac{R_e\{[1 + L_{d_i}(-j\omega)][1 + \rho^T\phi(j\omega)G_i(j\omega)]\}}{|1 + L_{d_i}(j\omega)|} < 0 \quad \forall\omega \quad \textit{for } i = 1, \dots, m \tag{22}$$

where $L_{d_i}(j\omega)$ is chosen such that the number of counterclockwise encirclement of the critical point by its Nyquist plot is equal to the number of unstable poles of $G_i(j\omega)$.

Proof: Since the real value of a complex number is less than or equal to its magnitude, we have:

$$R_e\{[1 + L_{d_i}(-j\omega)][1 + \rho^T\phi(j\omega)G_i(j\omega)]\} \leq |[1 + L_{d_i}(-j\omega)][1 + \rho^T\phi(j\omega)G_i(j\omega)]| \tag{23}$$

Then from (22) we obtain:

$$|W_{1i}(j\omega)| + |W_{2i}(j\omega)\rho^T\phi(j\omega)G_i(j\omega)| - |1 + \rho^T\phi(j\omega)G_i(j\omega)| < 0 \quad \forall\omega \quad \text{for } i = 1, \dots, m \tag{24}$$

which gives:

$$\left| \frac{W_{1i}(j\omega)}{1 + L_i(j\omega, \rho)} \right| + \left| \frac{W_{2i}(j\omega)L_i(j\omega, \rho)}{1 + L_i(j\omega, \rho)} \right| < 1 \quad \forall\omega \quad \text{for } i = 1, \dots, m \tag{25}$$

that leads directly to (21).

Now we should show that this controller stabilizes all models in $\mathcal{G}$. From (22), for $i = 1, \dots, m$, we have:

$$R_e\{[1 + L_{d_i}(-j\omega)][1 + \rho^T\phi(j\omega)G_i(j\omega)]\} > 0 \quad \forall\omega \tag{26}$$

or $\text{wno}\{[1 + L_{d_i}(-j\omega)][1 + L_i(j\omega, \rho)]\} = 0$, where wno stands for winding number around the origin. It should be mentioned that $L_{d_i}(-j\omega)$ and $L_i(j\omega, \rho)$ are zero or constant for the semicircle with infinity radius of the Nyquist contour so the wno depends only on the variation of s on the imaginary axis. Therefore:

$$\text{wno}[1 + L_{d_i}(j\omega)] = \text{wno}[1 + L_i(j\omega, \rho)] \tag{27}$$

Since $L_{d_i}(j\omega)$ satisfies the Nyquist criterion, $L_i(j\omega, \rho)$ will do as well and all closed-loop systems are stable. ■

Corollary 1 *Consider the convex combination of m spectral models in $\mathcal{G}$:*

$$\sum_{i=1}^{m} \lambda_i G_i(j\omega)[1 + W_{2i}(j\omega)\Delta] \triangleq G_\lambda(j\omega)[1 + W_2(j\omega)\Delta]$$

where

$$G_\lambda(j\omega) \triangleq \sum_{i=1}^{m} \lambda_i G_i(j\omega)$$

$$W_2(j\omega) \triangleq \frac{\sum_{i=1}^{m} \lambda_i G_i(j\omega) W_{2i}(j\omega)}{G_\lambda(j\omega)}$$

$\lambda = [\lambda_1, \dots, \lambda_m]$, $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda_i \in [0\,,\,1]$. Then, the linearly parameterized controller in (3) will stabilize this model for any admissible λ and satisfies the following robust performance condition:

$$\| |W_1\mathcal{S}| + |W_2\mathcal{T}| \|_\infty < 1 \tag{28}$$

where

$$W_1(j\omega) \triangleq \sum_{i=1}^{m} \lambda_i W_{1i}(j\omega)$$

if (22) is satisfied with $L_{d_i}(j\omega) = L_d(j\omega)$ for $i = 1, \dots, m$. $L_d(j\omega)$ should be chosen such that the number of counterclockwise encirclement of the critical point by its Nyquist plot is equal to the number of unstable poles of $G_\lambda(j\omega)$. A fixed $L_d(j\omega)$ means that the number of unstable poles of $G_\lambda(j\omega)$ should be fixed for all λ.

Proof: Multiplying (22) by λ_i and adding the m constraints we obtain:

$$\sum_{i=1}^{m} \lambda_i |W_{1i}(j\omega)| + \sum_{i=1}^{m} |W_{2i}(j\omega)\rho^T\phi(j\omega)\lambda_i G_i(j\omega)| - \frac{R_e\{[1 + L_d(-j\omega)][1 + \rho^T\phi(j\omega)\sum_{i=1}^{m} \lambda_i G_i(j\omega)]\}}{|1 + L_d(j\omega)|} < 0 \quad \forall\omega \tag{29}$$

We have : $|W_1(j\omega)| \leq \sum_{i=1}^{m} \lambda_i |W_{1i}(j\omega)|$ and

$$\left| \rho^T\phi(j\omega) \sum_{i=1}^{m} \lambda_i G_i(j\omega) W_{2i}(j\omega) \right| \leq \sum_{i=1}^{m} |W_{2i}(j\omega)\rho^T\phi(j\omega)\lambda_i G_i(j\omega)| \tag{30}$$

Therefore :

$$|W_1(j\omega)| + \left|\rho^T\phi(j\omega)G_\lambda(j\omega)W_2(j\omega)\right| - \frac{R_e\{[1+L_d(-j\omega)][1+\rho^T\phi(j\omega)G_\lambda(j\omega)]\}}{|1+L_d(j\omega)|} < 0 \qquad \forall\omega \tag{31}$$

The rest of the proof is similar to that of Theorem 1. ■

Remarks:

(1) The results of Theorem 1 are valid if $L_i(j\omega,\rho)$ has some poles on the imaginary axis, say $\{p_1, p_2, \ldots\}$. In this case $\omega \in \mathbb{R} - \{[p_1-\epsilon, p_1+\epsilon], [p_2-\epsilon, p_2+\epsilon], \ldots\}$ where ϵ is a small positive value. The stability is guaranteed if $L_{d_i}(j\omega)$ contains the poles on the imaginary axis of $L_i(j\omega,\rho)$ because they will have the same behavior at the small semicircular detour of the Nyquist contour at these poles.

(2) The same approach can be applied while an additive uncertainty model is available i.e.

$$\tilde{G}_i(s) = G_i(s) + W_{3i}(s)\Delta(s)$$

The robust performance condition is given by:

$$\left\| |W_{1i}\mathcal{S}_i| + \left|\frac{W_{3i}}{G_i}\mathcal{T}_i\right| \right\|_\infty < 1 \quad \text{for} \quad i = 1, \ldots, m \tag{32}$$

In this case the convex constraints in (22) can be used with the difference that

$$|W_{2i}(j\omega)| = |W_{3i}(j\omega)|/|G_i(j\omega)|$$

(3) Individual shaping of the sensitivity functions is also possible using the constraints in (22) with one of the filters equal to zero.

(4) The robust performance can be improved by minimizing the upper bound of the infinity norm of the weighted sensitivity function. Consider following optimization problem for a single model:

$$\min \gamma$$

$$\| |W_1\mathcal{S}| + |W_2\mathcal{T}| \|_\infty < \gamma \tag{33}$$

This optimization can be solved by an iterative bisection algorithm. At each iteration for a fixed γ_i, we replace W_1 and W_2 with W_1/γ_i and W_2/γ_i and we solve the feasibility problem represented by the linear constraints in (12) or convex constraints in (14). If the problem is feasible γ_{i+1} will be chosen smaller than γ_i and if the problem is infeasible γ_{i+1} will be increased.

3.3 How to deal with infinite number of constraints

It is shown in Theorem 1 that the problem of robust controller design for systems with multimodel and frequency-domain uncertainty can be formulated as a convex feasibility problem (or linear feasibility problem if we approximate the uncertainty circle by a polygon) with an infinite number of constraints. This problem is known as convex (or linear) semi-infinite program (SIP) for which different numerical solutions exist in the literature (see [22] for a survey).

A practical solution is to choose a finite number of frequencies and find a feasible solution for the constraints in (22) for $\omega \in \{\omega_1, \omega_2, \ldots, \omega_N\}$. It is clear that N should be sufficiently large such that the Nyquist diagram of $L(j\omega_k,\rho)$ is a good approximation of $L(j\omega,\rho)$.

For discrete-time controller design, since the frequency domain is limited to the half of sampling frequency, by increasing N the quality of approximation can be improved. This will increase the number of constraints but will not make a serious problem for linear programming methods which are able to deal with more than hundred thousands of linear constraints. For continuous-time controller design, the choice of N and the sampling frequency should be done cautiously. This will need some information about the plant and the desired closed-loop specifications.

If the spectral models are obtained from a set of noisy data, then the frequency-domain uncertainty sets are defined with a probability level. In this case, even a feasible solution to the semi infinite program will guarantee the robust performance with a probability level. Therefore, it is more reasonable to use a randomized approach to solve the SIP. According to the results of [23,24] with a reasonable number N of randomly chosen frequency samples, the optimal solution ρ^* to the convex optimization problem will satisfy the constraints for all frequencies with a high probability level. In order to be more precise, let the violation probability $V(\rho^*)$ be defined as the probability that for $\omega_0 \in \mathbb{R}$ the convex constraints are not satisfied for ρ^*. Then it can be shown that:

$$\mathcal{P}\{V(\rho^*) > \epsilon\} \le \sum_{i=0}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \tag{34}$$

where $\mathcal{P}\{\cdot\}$ stands for the probability of an event and ϵ is a satisfying level. Consider, for example, PID controller design ($n = 3$) with $N = 500$ frequency points. Then, having a violation probability of greater than $\epsilon = 0.01$ has a probability of less than 0.1234. This upper bound goes exponentially to zero with N. Therefore, the upper bound can be reduced to 0.0027 for $N = 1000$ and to 4.2×10^{-7} for $N = 2000$.

3.4 Choice of $L_d(s)$

It was shown in Proposition 1 that if the specifications are not too tight, for a large set of admissible $L_d(s)$ a nonempty inner convex approximation of the nonconvex set can be obtained. It was shown that a reasonable choice is $L_d(s)$ "close" to $L(s,\rho)$. Suppose that ρ^* is the optimal solution of the nonconvex problem. It is well known that the optimal H_∞ solution is based on cancellation of stable poles and zeros of the plant by the controller. Therefore, an $L_d(s)$ that contains the unstable poles and zeros of of the plant model and controller (including the poles on the imaginary axis) will be "close" to $L(s,\rho^*)$ and the convex set generated based on this $L_d(s)$ will likely contain the optimal controller. In this case the optimal controller can be found by minimizing a criterion $J(\rho) = \|L(\rho) - L_d\|$ under the robust performance constraint in (22). Since $L(\rho)$ is linear with respect to ρ any norm of $L(\rho) - L_d$ is a convex function of ρ.

For example, if we design a PID controller for open-loop stable systems with no pole on the imaginary axis a good choice is $L_d(s) = \omega_c/s$ with ω_c the desired closed-loop crossover frequency. It is clear that $L_d(s)$ contains only one integrator that reflects the integrator of the PID controller. This choice is coherent with the choice of desired open-loop transfer function in the classical open-loop shaping methods that suggest the magnitude of the open-loop transfer function should be large at low frequencies and small at high frequencies.

If the first choice of $L_d(j\omega)$ leads to a non feasible set, the iterative windsurfing approach [25] can be used to compute an appropriate $L_d(s)$. In this approach we start with modest specifications by reducing the gain of W_1 and W_2 so that a feasible solution ρ_1 is obtained. Then $L_d(j\omega) = L(j\omega, \rho_1)$ is chosen and the specifications will be tightened by increasing the gain of W_1 and W_2. A feasible solution ρ_2 for the second feasibility problem will be used to compute a new $L_d(j\omega) = L(j\omega, \rho_2)$. Although the convergence of this iterative approach to the optimal solution cannot be proved, good results in practice can be obtained.

The choice of $L_d(s)$ is more important for unstable systems. In this case, according to Theorem 1, the winding number of the Nyquist plot of $L_d(s)$ around the critical point should satisfy the Nyquist stability criterion. For this purpose, the number of unstable poles of the plant model should be known or a stabilizing controller $K_0(s)$ should be available. In the latter, $L_d(s) = K_0(s)G(s)$ is a good choice that satisfies the Nyquist criterion.

4 Simulation results

This example is taken from [26] where a robust performance problem is defined for an unstable plant. Consider the family of plants described by the following multiplicative uncertainty model:

$$\tilde{G}(s) = \frac{(s+1)(s+10)}{(s+2)(s+4)(s-1)}[1 + W_2(s)\Delta(s)] \quad (35)$$

where

$$W_2(s) = 0.8\frac{1.1337s^2 + 6.8857s + 9}{(s+1)(s+10)} \quad (36)$$

The nominal performance is defined by $\|W_1\mathcal{S}\|_\infty < 1$ with :

$$W_1(s) = \frac{2}{(20s+1)^2} \quad (37)$$

The objective is to compute a controller $K(s)$ that optimizes the robust performance by minimizing γ in (33).

The standard H_∞ solution that solves an approximate problem and leads to $\gamma_{\text{opt}} = 0.844$ for this problem with the controller $K(s) = N_\infty/D_\infty$, where

$$\begin{aligned} N_\infty &= 7.409e6s^6 + 1.266e8s^5 + 6.335e8s^4 + 1.152e9s^3 \\ &\quad +6.911e8s^2 + 5.442e7s + 9.37e5 \\ D_\infty &= s^7 + 9.07e5s^6 + 1.901e7s^5 + 1.043e8s^4 + 4.416e7s^3 \\ &\quad -4.682e7s^2 - 4.962e6s - 1.262e5 \end{aligned}$$

This 7th-order controller is unstable and has a pair of complex conjugate poles very close to the imaginary axis.

Now, the proposed method is applied to design a PID controller represented by :

$$K(s) = [K_p, K_i, K_d][1, \frac{1}{s}, \frac{s}{1+T_f s}]^T$$

where the time constant of the derivative part of the PID controller T_f is set to 0.01 s. The frequency response of the model is computed at $N = 500$ linearly spaced frequency points between 10^{-3} and 10^3 rad/s. The uncertainty circle at each frequency is approximated by an outbounding polygon with $q = 8$ vertices. The plant model contains one unstable pole and the controller an integrator, so the desired open-loop transfer function is chosen as

$$L_d(s) = \beta\frac{s+\alpha}{s(s-1)} \quad (38)$$

This is the simplest choice of $L_d(s)$ that contains a stable zero to ensure the Nyquist stability criterion. The characteristic polynomial of the closed-loop system with $L_d(s)$ is given by: $s^2 - s + \beta s + \beta\alpha$. Taking $\alpha = 1$ for simplicity, the stability criterion is satisfied for $L_d(s)$ with $\beta > 1$. For instance, we choose $\beta = 2$ and we will study later the sensitivity of the solution for different values of β.

In order to obtain the controller giving the minimal value for γ, the bisection algorithm explained in Remark 4 is used with the linear constraints in (12) that leads to

$$\||W_1\mathcal{S}| + |W_2\mathcal{T}|\|_\infty = 0.7262$$

The resulting PID controller is :

$$K_0(s) = \frac{2.074s^2 + 9.702s + 6.425}{0.01s^2 + s} \tag{39}$$

It is interesting to observe that this PID controller gives better performance than the H_∞ controller. Moreover, it is stable and easily implementable on a real system. The performance can be further improved using a new $L_d(s)$ based on $K_0(s)$. With this new $L_d(s) = K_0(s)G(s)$ the optimal controller is given by :

$$K(s) = \frac{2.643s^2 + 23.500s + 8.589}{0.01s^2 + s} \tag{40}$$

which leads to $\gamma_{\text{opt}} = 0.7247$.

In order to study the sensitivity of the solutions to the choice of $L_d(s)$, the value of β in (38) is changed from 2 to 97 with a step size of 5. For each value of β the minimum of γ is computed. The mean value of optimal γ's is 0.7611 and its standard deviation 0.0394. This shows that although the optimal solution depends on the choice of $L_d(s)$, it is not very sensitive to this choice. Moreover, the results obtained by this approach, whatever the choice of β between 2 and 97, are better than the standard H_∞ optimal solution.

5 Discussion and conclusions

It should be mentioned that the problem of robust fixed-order controller design is a non-convex NP-hard problem and all solutions to this problem, including ours, are based on some approximations. For example, if we consider the standard H_∞ control problem for design of fixed-order controllers for systems with multimodel and frequency-domain uncertainty, we have the following approximations :

- Approximation of the structured multimodel uncertainty with unstructured frequency-domain uncertainty.
- Approximation of the frequency-domain uncertainty with a rational weighting filter.
- Approximation of the real robust performance condition in (6) with the condition given in (7).
- Approximation of the resulting high-order controller with a fixed-order controller. In this operation, it is difficult to even guarantee the stability and performance for the reduced-order controller.

The proposed method considers directly the multimodel and frequency-domain uncertainty and designs a fixed-order controller. However, it seems that this method has some drawbacks which are discussed below :

(1) The proposed optimization problem has infinite number of constraints. However, in practice, a finite number of frequency points is sufficient for almost all applications.
(2) The controller is linearly parameterized so the denominator of the controller is fixed and it should be chosen prior to design. In practice, some of the poles of the controller are usually fixed to achieve certain closed-loop performances. For example a pole at origin, an integrator, or a pair of complex poles in a certain frequency are fixed in order to reject the disturbances (internal model principle). Therefore, this condition is not restrictive for low-order controller design. For higher order controller design the use of a set of orthogonal basis function is proposed. It is known that by increasing the controller order any stable transfer function can be approximated with such a set. On the other hand, this restriction ensures the stability of the controller which is required in many applications and cannot be guaranteed by a full controller parameterization. This means that this parameterization cannot be applied to systems which are not stabilizable by stable controllers.
(3) The robust performance condition in (6) is transformed to a set of linear constraints in (12) or convex constraints in (14). It is discussed in the paper that the conservatism of the approach depends on the choice of a desired open-loop transfer function.

It is too difficult (if not impossible) to compare, by a theoretical analysis, the overall approximation or conservatism of different approaches to fixed-order controller design. In this paper we tried to show the effectiveness of the proposed approach by means of a simulation example. This approach has been applied to an international benchmark problem for robust controller design [27] and a controller with only 7 parameters has been designed that meets all benchmark specifications. These results are not included in this paper because of space limitation but are available in [28]. The advantages of this approach are summarized below:

- The method uses only the frequency response of the system and no parametric model is required. The frequency response of the model and the uncertainty at each frequency can be obtained directly by discrete Fourier transform from a set of data, so the method can be considered as completely "data-driven". Of course, the method can be applied as well if a parametric model with a pure time delay and an uncertainty set is available.
- The method is very simple, at least as simple as open-loop shaping methods in Bode diagram or in Nichols

chart currently used in textbooks for undergraduate courses in control systems. For instance, it can be used to design of PID controllers ensuring a given modulus margin and optimizing for a desired crossover frequency by a quadratic programming optimization approach. Moreover, the case of multimodel uncertainty can be handled easily just by increasing the number of linear constraints while the mentioned classical frequency-domain approaches cannot deal with this type of uncertainty.
- Higher order controllers for unstable systems with H_∞ type specifications can also be designed within the same framework.
- Although only SISO systems are discussed in this paper, the extension to MIMO systems is also possible thanks to Gershgorin bands. In the same framework, multivariable controllers can be designed that decouple the off-diagonal elements of the open-loop transfer matrix and meets the H_∞ specifications for the decoupled system [29].

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