

# CHARACTERIZATION OF THE TRACES ON THE BOUNDARY OF FUNCTIONS IN MAGNETIC SOBOLEV SPACES

HOAI-MINH NGUYEN AND JEAN VAN SCHAFTINGEN

ABSTRACT. We characterize the trace of magnetic Sobolev spaces defined in a half-space or in a smooth bounded domain in which the magnetic field  $A$  is differentiable and its exterior derivative corresponding to the magnetic field  $dA$  is bounded. In particular, we prove that, for  $d \geq 1$  and  $p > 1$ , the trace of the magnetic Sobolev space  $W_A^{1,p}(\mathbb{R}_+^{d+1})$  is exactly  $W_{A''}^{1-1/p,p}(\mathbb{R}^d)$  where  $A''(x) = (A_1, \dots, A_d)(x, 0)$  for  $x \in \mathbb{R}^d$  with the convention  $A = (A_1, \dots, A_{d+1})$  when  $A \in C^1(\mathbb{R}_+^{d+1}, \mathbb{R}^{d+1})$ . We also characterize fractional magnetic Sobolev spaces as interpolation spaces and give extension theorems from a half-space to the entire space.

## 1. INTRODUCTION

The *first-order magnetic Sobolev space*  $W_A^{1,p}(\Omega)$  on a given open set  $\Omega \subset \mathbb{R}^{d+1}$  with  $d \geq 1$  is defined, for a given exponent  $p \in [1, +\infty)$ , a vector field  $A \in C^1(\Omega, \mathbb{R}^{d+1})$ , as [3; 4; 6; 11; 16; 19; 28; 29, §1.1]

$$(1.1) \quad W_A^{1,p}(\Omega) \triangleq \left\{ U \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{C}); \|U\|_{W_A^{1,p}(\Omega)}^p \triangleq \int_{\Omega} |U|^p + |\nabla_A U|^p < +\infty \right\},$$

where the weak covariant gradient  $\nabla_A U$  associated with  $A$  of  $U \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{C})$  is defined as <sup>1</sup>

$$(1.2) \quad \nabla_A U = \nabla U + iAU \quad \text{in } \Omega.$$

Magnetic Sobolev spaces arise naturally for  $p = 2$  and  $d = 2$  (corresponding to  $\Omega \subseteq \mathbb{R}^3$ ) in *quantum mechanics* in the presence of a magnetic field described through its *magnetic vector potential*  $A \in C^1(\Omega, \mathbb{R}^3)$ ; the function  $U : \Omega \rightarrow \mathbb{C}$  is then a wave-function and the integral in (1.1) is the quadratic form associated to the quantum mechanical Hamiltonian of a particle in a magnetic field (see, e.g., [14, Chapter 16; 17, Chapter XV]). In physical models, the only observable quantities are the *magnetic field*  $B = \nabla \times A \simeq dA \in C(\Omega, \wedge^2 \mathbb{R}^3)$  and the *probability density*  $|U|^2$ . Here and in what follows,  $dA$  denotes the exterior derivative of  $A$ ; for this, we consider  $A$  as an element in  $C^1(\Omega, \wedge^1 \mathbb{R}^{d+1})$ . The prevalent role of the magnetic field and of the probability density is reflected by the *gauge invariance* invariance of the model: the invariance of the Hamiltonian quadratic form defined by the left-hand side (1.1) under a change of variables  $A \mapsto A + \nabla \Phi$  and  $U \mapsto e^{-i\Phi} U$ , for any phase shift  $\Phi \in C^1(\Omega, \mathbb{R})$ , see, e.g.,

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(H.-M. Nguyen) ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL), SB, CAMA, STATION 8, CH-1015 LAUSANNE, SWITZERLAND

(J. Van Schaftingen) UNIVERSITÉ CATHOLIQUE DE LOUVAIN (UCLouvain), INSTITUT DE RECHERCHE EN MATHÉMATIQUE ET PHYSIQUE, CHEMIN DU CYCLOTRON 2 BTE L7.01.01, 1348 LOUVAIN-LA-NEUVE, BELGIUM  
*E-mail addresses:* hoai-minh.nguyen@epfl.ch, Jean.VanSchaftingen@uclouvain.be.

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<sup>1</sup>Two opposite conventions are in use for the sign of the second term in (1.2), we follow [3, 11, 19, 28], and we have thus opposite convention to [4; 6; 16; 29, §1.1]. The presence of two opposite conventions is related to charge of the particles that are studied; both conventions are equivalent up to complex conjugation.

[19, chapter 7]. Geometrically, the invariant quantity  $i dA$  is the curvature of the associated  $U(1)$ -connection (see for example [30, Chapter 11]).

Magnetic Sobolev spaces  $W_A^{1,p}(\Omega)$  generalize *classical Sobolev spaces*  $W^{1,p}(\Omega)$ , in which  $A \equiv 0$ , defined by

$$W^{1,p}(\Omega) \triangleq \left\{ U \in L^p(\Omega); \|U\|_{W^{1,p}(\Omega)}^p \triangleq \int_{\Omega} |U|^p + |\nabla U|^p < +\infty \right\}.$$

For  $0 < s < 1$  and  $1 \leq p < +\infty$ , the *fractional Sobolev (Sobolev–Slobodeckii) space* is defined as

$$W^{s,p}(\partial\Omega) \triangleq \left\{ u \in L^p(\partial\Omega, \mathbb{C}); \|u\|_{W^{s,p}(\partial\Omega)}^p \triangleq \|u\|_{L^p(\partial\Omega)}^p + |u|_{W^{s,p}(\partial\Omega)}^p < +\infty \right\},$$

where the *Gagliardo seminorm*  $|u|_{W^{s,p}(\partial\Omega)}$  of the function  $u : \partial\Omega \rightarrow \mathbb{C}$  is given by

$$|u|_{W^{s,p}(\partial\Omega)}^p \triangleq \iint_{\partial\Omega \times \partial\Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{d+sp}} dx dy.$$

When the set  $\Omega$  is bounded and its boundary is of class  $C^1$ , or  $\Omega = \mathbb{R}_+^{d+1} \triangleq \{(x, t) \in \mathbb{R}^d \times \mathbb{R}; t > 0\}$ , and when  $p > 1$ , the trace theory is well known since Gagliardo's pioneer work [13] (see also [9, §10.17–10.18 and Proposition 17.1; 22; 34]). The trace operator  $\text{Tr}$  defined by

$$\text{Tr} : C^1(\bar{\Omega}) \rightarrow C^1(\partial\Omega)$$

$$U \mapsto U|_{\partial\Omega},$$

satisfies for some positive constant  $C_{p,\Omega}$ , for every  $U \in C^1(\Omega)$  the estimate,

$$\|\text{Tr} U\|_{W^{1-1/p,p}(\partial\Omega)} \leq C_{p,\Omega} \|U\|_{W^{1,p}(\Omega)},$$

and therefore the linear operator  $\text{Tr}$  extends to a bounded linear map from the Sobolev space  $W^{1,p}(\Omega)$  into fractional Sobolev space  $W^{1-1/p,p}(\partial\Omega)$ . Conversely there exists a bounded linear operator  $\text{Ext} : W^{1-1/p,p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$  such that for any  $u \in W^{1-1/p,p}(\partial\Omega)$ ,

$$\text{Tr}(\text{Ext } u) = u \text{ on } \partial\Omega \quad \text{and} \quad \|\text{Ext } u\|_{W^{1,p}(\Omega)} \leq C'_{p,\Omega} \|u\|_{W^{1-1/p,p}(\partial\Omega)},$$

for some positive constant  $C'_{p,\Omega}$  independent of  $u$ . In particular, the map  $\text{Tr}$  is surjective. Consequently, the image under the trace operator of the space  $W^{1,p}(\Omega)$  is exactly the space  $W^{1-1/p,p}(\partial\Omega)$ . The space of traces can also be described as the real interpolations space  $(W^{1,p}(\mathbb{R}^d), L^p(\mathbb{R}^d))_{1-1/p,p}$  in the framework of interpolation of Banach spaces [20, Théorème VI.2.1].

The trace theory for  $W_A^{1,p}(\Omega)$  can easily be derived from the one of  $W^{1,p}(\Omega)$  when the magnetic potential  $A$  is bounded. In fact, by the triangle inequality

$$(1.3) \quad \left| \|\nabla_A u\|_{L^p(\Omega)} - \|\nabla u\|_{L^p(\Omega)} \right| \leq \|A\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega)},$$

it follows that  $W_A^{1,p}(\Omega) = W^{1,p}(\Omega)$  in this case. Hence the trace of  $W_A^{1,p}(\Omega)$  is the space  $W^{1-1/p,p}(\partial\Omega)$  as well. The situation becomes more delicate when  $\Omega = \mathbb{R}_+^{d+1}$  and  $A$  is not assumed to be bounded but its total derivative  $DA$  or, even more physically, its exterior derivative  $dA$  is bounded. This type of assumption on  $A$  appears naturally in many problems in physics for which  $A$  is linear in simple settings. Moreover, even when  $A$  is bounded, the quantitative bounds resulting (1.3) depend on  $\|A\|_{L^\infty(\Omega)}$  which is not gauge invariant; it would be desirable to have estimates depending rather on  $dA$ . To our knowledge, a characterization of the trace of  $W_A^{1,p}(\mathbb{R}_+^{d+1})$  is not known under such assumption on  $A$ . The goal of this work is to give a complete answer to this question. Besides its own interest concerning boundary values in problems of calculus of variations and partial differential equations, this is closely related to classes of fractional magnetic problems motivated by relativistic magnetic quantum physical models [15] that have been studied recently [1, 2, 5, 8, 12, 18, 25–27, 31, 35].

Given  $0 < s < 1$ ,  $1 \leq p < +\infty$ , and  $A'' \in C(\mathbb{R}^d, \mathbb{R}^d)$ , we define, for any measurable function  $u : \mathbb{R}^d \rightarrow \mathbb{C}$

$$(1.4) \quad |u|_{W_{A''}^{s,p}(\mathbb{R}^d, \mathbb{C})}^p \triangleq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy,$$

where the potential  $\mathcal{I}_{A''} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is defined for each  $x, y \in \mathbb{R}^d$  by

$$\mathcal{I}_{A''}(x, y) \triangleq \int_0^1 A''((1-t)x + ty) \cdot (y-x) dt.$$

Here and in what follows  $\cdot$  denotes the complex scalar product. For  $A = (A_1, \dots, A_{d+1}) \in C(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$ , we will consider its parallel component on the boundary  $A'' : \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined for each  $x \in \mathbb{R}^d$  by

$$(1.5) \quad A''(x) = (A_1, \dots, A_d)(x, 0).$$

Our first main result is

**Theorem 1.1.** *Let  $d \geq 1$  and  $1 < p < +\infty$ . There exists a positive constant  $C_{d,p}$  depending only on  $d$  and  $p$  such that if  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$  and  $\|dA\|_{L^\infty(\mathbb{R}_+^{d+1})} \leq \beta$ , then*

(i) *for each  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{C})$ ,*

$$|U(\cdot, 0)|_{W_{A''}^{1-1/p,p}(\mathbb{R}^d)} + \beta^{\frac{1}{2} - \frac{1}{2p}} \|U(\cdot, 0)\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \left( \|\nabla_A U\|_{L^p(\mathbb{R}_+^{d+1})} + \beta^{\frac{1}{2}} \|U\|_{L^p(\mathbb{R}_+^{d+1})} \right),$$

(ii) *for each  $u \in C_c^1(\mathbb{R}^d, \mathbb{C})$ , there exists  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}})$  depending linearly on  $u$  such that  $U(x, 0) = u(x)$  in  $\mathbb{R}^d$  and*

$$\|\nabla_A U\|_{L^p(\mathbb{R}_+^{d+1})} + \beta^{\frac{1}{2}} \|U\|_{L^p(\mathbb{R}_+^{d+1})} \leq C_{d,p} \left( |u|_{W_{A''}^{1-1/p,p}(\mathbb{R}^d)} + \beta^{\frac{1}{2} - \frac{1}{2p}} \|u\|_{L^p(\mathbb{R}^d)} \right).$$

The conclusions of Theorem 1.1 are *gauge-invariant*: all the functional norms are gauge-invariant and the constants only depend through  $\beta$  which is an upper bound of the norm  $\|dA\|_{L^\infty(\mathbb{R}_+^{d+1})}$  of the magnetic field on the half-space  $\mathbb{R}_+^{d+1}$ .

As a consequence of Theorem 1.1, by a standard density argument (see Section 4), we obtain the following characterization of the trace of the space  $W_A^{1,p}(\mathbb{R}_+^{d+1})$ :

**Theorem 1.2.** *Let  $d \geq 1$  and  $1 < p < +\infty$ . Assume that  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \wedge^1 \mathbb{R}^{d+1})$  and that  $dA \in L^\infty(\mathbb{R}_+^{d+1}, \wedge^2 \mathbb{R}^{d+1})$ . The trace mapping*

$$\text{Tr} : W_A^{1,p}(\mathbb{R}_+^{d+1}, \mathbb{C}) \rightarrow W_{A''}^{1-1/p,p}(\mathbb{R}^d, \mathbb{C})$$

$$U(x, x_{d+1}) \mapsto U(x, 0)$$

*is linear and continuous. There exists a linear continuous mapping*

$$\text{Ext} : W_{A''}^{1-1/p,p}(\mathbb{R}^d, \mathbb{C}) \rightarrow W_A^{1,p}(\mathbb{R}_+^{d+1}, \mathbb{C})$$

*such that  $\text{Tr} \circ \text{Ext}_{\mathbb{R}_+^{d+1}}$  is the identity on  $W_{A''}^{1-1/p,p}(\mathbb{R}^d)$ . Moreover, the corresponding estimates of Theorem 1.1 with  $u = \text{Tr} U$  and  $U = \text{Ext} u$  are valid.*

In the case where the magnetic field  $dA$  is constant, we obtain the following improvements:

**Theorem 1.3.** *Let  $d \geq 1$  and  $1 < p < +\infty$ . Assume that  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$  and that  $dA$  is constant. We have, with  $u = \text{Tr} U$  and  $U = \text{Ext} u$ ,*

$$(1.6) \quad |u|_{W_{A''}^{1-1/p,p}(\mathbb{R}^d)} + \|dA\|^{\frac{1}{2} - \frac{1}{2p}} \|u\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla_A U\|_{L^p(\mathbb{R}_+^{d+1})}$$

and

$$(1.7) \quad \|\nabla_A U\|_{L^p(\mathbb{R}_+^{d+1})} + \|dA\|^{\frac{1}{2}} \|U\|_{L^p(\mathbb{R}_+^{d+1})} \leq C_{d,p} |u|_{W_{A''}^{1-1/p,p}(\mathbb{R}^d)}.$$

We later show that the space  $W_{A^u}^{s,p}(\mathbb{R}^d)$  with  $0 < s < 1$  and  $p \geq 1$  is the trace space of the space  $W_{A,1-(1-s)p}^{1,p}(\mathbb{R}^d)$  whose definition is given in (4.1); moreover, the corresponding estimates hold (see Theorems 4.3 and 5.4).

We establish similar estimates for a smooth bounded domain  $\Omega \subset \mathbb{R}^{d+1}$  and a magnetic potential  $A \in C^1(\bar{\Omega}, \mathbb{R}^{d+1})$ . It is worth noting that the trace theory in this setting is known as in the case  $A \equiv 0$ . Nevertheless, our estimates (Proposition 6.4 and Proposition 6.5) are gauge invariant, and sharpen estimates in the semi-classical limit (Proposition 6.6).

As a consequence of the trace theorems, we derive a characterization of the space  $W_{A^u}^{s,p}(\mathbb{R}^d, \mathbb{C})$  as an interpolation space (Theorem 7.2). We also observe that the characterization of traces is also independent on the side of the hyperplane from which the trace is taken or to which the extension is made (this fact is not too trivial, see Remark 8.2). Consequently, the trace theorem provides an extension theorem from a half-space to the whole space (Theorem 8.1).

In an appendix, we show that under the assumption that some derivative of  $A$  is bounded, our magnetic fractional spaces have equivalent norms to other families of fractional spaces defined in the literature (Proposition A.1).

We now describe briefly the idea of the proof of the trace theory. The proof of the trace estimates and of the construction of the extension is based on a standard strategy that goes back to Gagliardo's seminal work [13]. Concerning Theorem 1.1 and its variants (Propositions 2.1 and 3.1), the key point of our analysis lies on the observation that  $A^u$  defined in  $\mathbb{R}^d$  by (1.5) encodes the information the trace space of  $W_A^{1,p}(\mathbb{R}_+^{d+1})$  and an appropriate extension formula given in (3.1). The proof of the trace estimates also involves Stokes theorems (Lemma 2.2) and a simple useful observation given in Lemma 2.3. Concerning Theorem 1.3 and its variants (Theorem 5.4), the new part is the trace estimates (see, e.g., (1.6)). To this end, the Stoke formula and an averaging argument are used while taking into account the fact  $dA$  is constant. The proof for a domain  $\Omega$  uses the results in the half space via local charts.

## 2. TRACE ESTIMATE FOR BOUNDED MAGNETIC FIELD

In this section, we prove the following trace estimate on the boundary of the half-space with a bounded magnetic field, which covers (i) in Theorem 1.1.

**Proposition 2.1.** *Let  $d \geq 1$ ,  $0 < s < 1$ , and  $1 \leq p < +\infty$ . There exists a positive constant  $C_{d,s,p}$  depending only on  $d$ ,  $s$  and  $p$  such that if  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$ ,  $\|dA\|_{L^\infty(\mathbb{R}_+^{d+1})} \leq \beta$  and if  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{C})$ , then*

$$(2.1) \quad |U(\cdot, 0)|_{W_{A^u}^{s,p}(\mathbb{R}^d)}^p \leq C_{d,s,p} \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z, t)|^p + \beta^{\frac{p}{2}} |U(z, t)|^p}{t^{1-(1-s)p}} dz dt$$

and

$$(2.2) \quad \|U(\cdot, 0)\|_{L^p(\mathbb{R}^d)}^p \leq C_{d,s,p} \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z, t)|^p}{t^{1-(1-s)p}} dt dz \right)^{1-s} \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|U(z, t)|^p}{t^{1-(1-s)p}} dt dz \right)^s.$$

As a consequence of (2.2), we have

$$(2.3) \quad \beta^{\frac{sp}{2}} \|U(\cdot, 0)\|_{L^p(\mathbb{R}^d)}^p \leq C' \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z, t)|^p + \beta^{\frac{p}{2}} |U(z, t)|^p}{t^{1-(1-s)p}} dz dt.$$

We first present several lemmas used in the proof of Proposition 2.1, before going to the proof of Proposition 2.1 at the end of the section.

We define for each  $X, Y \in \overline{\mathbb{R}_+^{d+1}}$ , the homotopy operator

$$(2.4) \quad \mathcal{I}_A(X, Y) \triangleq \int_0^1 A((1-t)X + tY) \cdot (Y - X) dt.$$

We observe by integration by parts that

$$(2.5) \quad D\mathcal{I}_A(\cdot, Y) = -A.$$

Therefore, by the fundamental theorem of calculus, we have

$$(2.6) \quad e^{i\mathcal{I}_A(X,Y)}U(Y) - U(X) = \int_0^1 e^{i\mathcal{I}_A(X,(1-t)X+tY)}\nabla_A U((1-t)X+tY) \cdot (Y-X) dt.$$

The following result will be used repeatedly in the present work.

**Lemma 2.2.** *If  $d \geq 1$ ,  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$ , then for every  $X, Y, Z \in \overline{\mathbb{R}_+^{d+1}}$ , we have*

$$\mathcal{I}_A(X, Y) + \mathcal{I}_A(Y, Z) + \mathcal{I}_A(Z, X) = \int_0^1 \int_0^{1-s} dA((1-t-s)X+tY+sZ)[Y-X, Z-X] dt ds.$$

*Proof.* We have if  $s, t \in [0, 1]$  and  $s+t \leq 1$ ,

$$\begin{aligned} & dA((1-t-s)X+tY+sZ)[Y-X, Z-X] \\ &= \frac{d}{dt}A((1-t-s)X+tY+sZ)[Z-X] - \frac{d}{ds}A((1-t-s)X+tY+sZ)[Y-X]. \end{aligned}$$

Integrating with respect to  $s, t \in [0, 1]$  with  $s+t \leq 1$  yields

$$(2.7) \quad \begin{aligned} & \int_0^1 \int_0^{1-s} dA((1-t-s)X+tY+sZ)[Y-X, Z-X] dt ds \\ &= \int_0^1 \int_0^{1-s} \frac{d}{dt}A((1-t-s)X+tY+sZ)[Z-X] dt ds \\ & \quad - \int_0^1 \int_0^{1-t} \frac{d}{ds}A((1-t-s)X+tY+sZ)[Y-X] ds dt. \end{aligned}$$

By the fundamental theorem of calculus, for every  $s \in [0, 1]$  we have

$$(2.8) \quad \begin{aligned} & \int_0^{1-s} \frac{d}{dt}A((1-t-s)X+tY+sZ)[Z-X] dt \\ &= A((1-s)Y+sZ)[Z-X] - A((1-s)X+sZ)[Z-X] \end{aligned}$$

and for every  $t \in [0, 1]$ ,

$$(2.9) \quad \begin{aligned} & \int_0^{1-t} \frac{d}{ds}A((1-t-s)X+tY+sZ)[Y-X] ds \\ &= A(tY+(1-t)Z)[Y-X] - A((1-t)X+tY)[Y-X]. \end{aligned}$$

By inserting (2.8), (2.9) and (2.7) and by applying the change of variable  $s=t$ ,  $t=1-s$ , we obtain

$$\begin{aligned} & \int_0^1 \int_0^{1-s} dA((1-t-s)X+tY+sZ)[Y-X, Z-X] dt ds \\ &= \int_0^1 A((1-s)Y+sZ)[Z-X] ds - \int_0^1 A(tX+(1-t)Z)[Z-X] dt \\ & \quad - \int_0^1 A((1-s)Y+sZ)[Y-X] ds + \int_0^1 A((1-t)X+tY)[Y-X] dt \\ &= \mathcal{I}_A(Z, X) + \mathcal{I}_A(Y, Z) + \mathcal{I}_A(X, Y), \end{aligned}$$

in view of the definition in (2.4). □

Using Lemma 2.2, we can establish the following simple result which is the key ingredient of the proof of Proposition 2.1.

**Lemma 2.3.** *If  $d \geq 1$ ,  $U \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{C})$  and  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$ , then for every  $X, Y, Z \in \overline{\mathbb{R}_+^{d+1}}$ , we have*

$$|e^{i\mathcal{I}_A(X,Y)}U(Y) - U(X)| \leq |e^{i\mathcal{I}_A(Z,Y)}U(Y) - U(Z)| + |e^{i\mathcal{I}_A(Z,X)}U(X) - U(Z)| \\ + |U(Z)| \min \left\{ 1, \frac{1}{2} \|dA\|_{L^\infty} |(X - Z) \wedge (Y - Z)| \right\}.$$

*Proof.* Since

$$|e^{i\mathcal{I}_A(X,Y)}U(Y) - e^{i(\mathcal{I}_A(X,Y)+\mathcal{I}_A(Y,Z))}U(Z)| = |e^{i\mathcal{I}_A(Z,Y)}U(Y) - U(Z)|,$$

and

$$|U(X) - e^{i\mathcal{I}_A(X,Z)}U(Z)| = |e^{i\mathcal{I}_A(Z,X)}U(X) - U(Z)|,$$

by the triangle inequality, we obtain

$$|e^{i\mathcal{I}_A(X,Y)}U(Y) - U(X)| \leq |e^{i\mathcal{I}_A(Z,Y)}U(Y) - U(Z)| + |e^{i\mathcal{I}_A(Z,X)}U(X) - U(Z)| \\ + |e^{i(\mathcal{I}_A(X,Y)+\mathcal{I}_A(Y,Z))}U(Z) - e^{i\mathcal{I}_A(X,Z)}U(Z)|.$$

We observe that

$$|e^{i(\mathcal{I}_A(X,Y)+\mathcal{I}_A(Y,Z))}U(Z) - e^{i\mathcal{I}_A(X,Z)}U(Z)| = |e^{i(\mathcal{I}_A(X,Y)+\mathcal{I}_A(Y,Z)+\mathcal{I}_A(Z,X))} - 1||U(Z)|.$$

and conclude with Lemma 2.2.  $\square$

*Proof of Proposition 2.1.* Applying Lemma 2.3, we have, for each  $x, y \in \mathbb{R}^d$  and with the notations  $x \simeq (x, 0) \in \mathbb{R}_+^{d+1}$ ,  $y \simeq (y, 0) \in \mathbb{R}_+^{d+1}$ ,  $Z = (\frac{x+y}{2}, |y-x|)$  and  $u = U(\cdot, 0)$ ,

$$(2.10) \quad |e^{i\mathcal{I}_A(x,y)}u(y) - u(x)| \\ \leq |e^{i\mathcal{I}_A(Z,x)}U(x) - U(Z)| + |e^{i\mathcal{I}_A(Z,y)}U(y) - U(Z)| + |U(Z)| \|dA\|_{L^\infty}^{1/2} |y-x|.$$

Using (2.6), we derive from (2.10) that

$$(2.11) \quad |e^{i\mathcal{I}_A(x,y)}u(y) - u(x)| \leq |y-x| \int_0^1 |\nabla_A U((1-t)x + tZ)| dt \\ + |y-x| \int_0^1 |\nabla_A U((1-t)y + tZ)| dt + C_1 \beta^{\frac{1}{2}} |y-x| |U(Z)|.$$

Here and in what follows in this proof,  $C_1, C_2, \dots$  denote positive constants depending only on  $d, p$ , and  $s$ .

Since  $0 < s < 1$ , using the fact, for a measurable function  $f$  defined on  $[0, 1]$ , by Hölder's inequality, that

$$\left( \int_0^1 |f(t)| dt \right)^p \leq C_2 \int_0^1 t^{(1-s)(p-1)} |f(t)|^p dt,$$

we derive from (2.11) that, since  $Z = (\frac{x+y}{2}, |y-x|)$ ,

$$(2.12) \quad \frac{|e^{i\mathcal{I}_A(x,y)}U(y) - U(x)|}{|y-x|^{d+sp}} \leq C_3 \left( \int_0^1 t^{(1-s)(p-1)} \frac{|\nabla_A U((1-\frac{t}{2})x + \frac{t}{2}y, t|y-x|)|^p}{|y-x|^{d-(1-s)p}} dt \right. \\ \left. + \int_0^1 t^{(1-s)(p-1)} \frac{|\nabla_A U((1-\frac{t}{2})y + \frac{t}{2}x, t|y-x|)|^p}{|y-x|^{d-(1-s)p}} dt \right. \\ \left. + \beta^{\frac{p}{2}} \frac{|U(\frac{x+y}{2}, |y-x|)|^p}{|y-x|^{d-(1-s)p}} \right).$$

We now estimate the integral of the left-hand side of (2.12) with respect to  $x$  and  $y$  by estimating the integrals of the three terms on the right-hand side. For every  $t \in (0, 1)$ , making the change of variable  $\eta = (1 - \frac{t}{2})x + \frac{t}{2}y$  and  $\xi = t(x - y)$ , we obtain for every  $t \in (0, 1)$ ,

$$(2.13) \quad t^{(1-s)(p-1)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A(U(1 - \frac{t}{2})x + \frac{t}{2}y, t|y - x|)|^p}{|y - x|^{d-(1-s)p}} dx dy \\ = \frac{1}{t^{1-s}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U(\eta, |\xi|)|^p}{|\xi|^{d-(1-s)p}} d\eta d\xi = \frac{|\mathbb{S}^{d-1}|}{t^{1-s}} \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|\nabla_A U(\eta, r)|^p}{r^{1-(1-s)p}} d\eta dr.$$

Since  $0 < s < 1$ , it follows that

$$(2.14) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 t^{(1-s)(p-1)} \frac{|\nabla_A(U(1 - \frac{t}{2})x + \frac{t}{2}y, t|y - x|)|^p}{|y - x|^{d-(1-s)p}} dt dx dy \\ \leq C_4 \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|\nabla_A U(x, r)|^p}{r^{1-(1-s)p}} dx dr.$$

Similarly, we have

$$(2.15) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 t^{(1-s)(p-1)} \frac{|\nabla_A(U(1 - \frac{t}{2})y + \frac{t}{2}x, t|y - x|)|^p}{|y - x|^{d-(1-s)p}} dt dx dy \\ \leq C_4 \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|\nabla_A U(x, r)|^p}{r^{1-(1-s)p}} dx dr.$$

Using the change of variable  $\eta = \frac{x+y}{2}$  and  $\xi = y - x$ , and the polar coordinates, by the same way to obtain (2.13), we also reach

$$(2.16) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U(\frac{x+y}{2}, |y - x|)|^p}{|x - y|^{d-(1-s)p}} dx dy = |\mathbb{S}^{d-1}| \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|U(\eta, r)|^p}{r^{1-(1-s)p}} d\eta dr.$$

Combining (2.12), (2.14), (2.15), and (2.16) yields

$$(2.17) \quad |u|_{W_{A^u}^{s,p}(\mathbb{R}^d)}^p \leq C_5 \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z, t)|^p + \|dA\|_{L^\infty}^{\frac{p}{2}} |U(z, t)|^p}{t^{1-(1-s)p}} dz dt,$$

which is (2.1).

We next prove (2.2). Since, by the diamagnetic inequality  $|\nabla|U|| \leq |\nabla_A U|$  in  $\mathbb{R}_+^{d+1}$ , we have, for  $x \in \mathbb{R}^d$  and  $t \geq 0$ ,

$$|U(x, 0)| \leq |U(x, t)| + \int_0^t |\nabla_A U(x, s)| ds.$$

It follows that, for  $x \in \mathbb{R}^d$  and  $\lambda > 0$ ,

$$|U(x, 0)| \leq \int_0^\lambda |\nabla_A U(x, t)| dt + \frac{2}{\lambda} \int_{\lambda/2}^\lambda |U(x, t)| dt.$$

Using Hölder's inequality, we deduce that

$$(2.18) \quad \begin{aligned} \left( \int_{\mathbb{R}^d} |U(x, 0)|^p dx \right)^{\frac{1}{p}} &\leq \int_0^\lambda \left( \int_{\mathbb{R}^d} |\nabla_A U(x, t)|^p dx \right)^{\frac{1}{p}} dt + \frac{2}{\lambda} \int_{\frac{\lambda}{2}}^\lambda \left( \int_{\mathbb{R}^d} |U(x, t)|^p dx \right)^{\frac{1}{p}} dt \\ &\leq C_6 \lambda^s \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x, t)|^p}{t^{1-(1-s)p}} dx dt \right)^{\frac{1}{p}} + \frac{C_7}{\lambda^{1-s}} \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|U(x, t)|^p}{t^{1-(1-s)p}} dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

Optimizing with respect to  $\lambda > 0$ , we obtain

$$(2.19) \quad \left( \int_{\mathbb{R}^d} |U(x, 0)|^p dx \right)^{\frac{1}{p}} \leq C_8 \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x, t)|^p}{t^{1-(1-s)p}} dx dt \right)^{\frac{1-s}{p}} \left( \iint_{\mathbb{R}_+^{d+1}} \frac{|U(x, t)|^p}{t^{1-(1-s)p}} dx dt \right)^{\frac{s}{p}},$$

which is (2.2).  $\square$

In what follows  $B(x, R)$  denotes the open ball in  $\mathbb{R}^d$  centered at  $x$  and of radius  $R$ ; when  $x = 0$ , one uses the notation  $B_R$  instead. Using the same arguments, we obtain a localized version of Proposition 2.1 which will be used in Section 6.

**Proposition 2.4.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . There exists a positive constant  $C_{d,s,p}$  depending only on  $s, p$ , and  $d$  such that if  $R \in (0, +\infty)$ ,  $A \in C^1(B(0, R) \times [0, R], \mathbb{R}^{d+1})$ , if  $\|dA\|_{L^\infty} \leq \beta$  and if  $U \in C^\infty(B[0, R] \cap \mathbb{R}_+^{d+1})$  and  $u = U(\cdot, 0)$ , then*

$$\iint_{B(0,R) \times B(0,R)} \frac{|e^{i\mathcal{I}_{A^u}(x,y)} U(y, 0) - U(x, 0)|^p}{|y - x|^{d+sp}} dx dy \leq C_{d,s,p} \iint_{B(0,R) \times [0,R]} \frac{|\nabla_A U(z, t)|^p + \beta^{\frac{p}{2}} |U(z, t)|^p}{t^{1-(1-s)p}} dt dz.$$

### 3. EXTENSION TO THE HALF-SPACE

In this section, we prove the following extension result which implies (ii) of Theorem 1.1.

**Proposition 3.1.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . There exists a positive constant  $C_{d,s,p}$  depending only on  $d, s$ , and  $p$  such that for every  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$  with  $\|dA\|_{L^\infty(\mathbb{R}_+^{d+1})} \leq \beta$  and for any  $u \in C_c^1(\mathbb{R}^d, \mathbb{C})$  with compact support, one can find  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}})$  depending linearly on  $u$  such that  $U(x, 0) = u(x)$  in  $\mathbb{R}^d$ ,*

$$\iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x, t)|^p}{t^{1-(1-s)p}} dx dt \leq C_{d,s,p} \left( \|u\|_{W_{A^u}^{s,p}(\mathbb{R}^d)}^p + \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p \right)$$

and

$$\iint_{\mathbb{R}_+^{d+1}} \frac{|U(z, t)|^p}{t^{1-(1-s)p}} dx dt \leq \frac{C_{d,s,p}}{\beta^{\frac{(1-s)p}{2}}} \|u\|_{L^p(\mathbb{R}^d)}^p.$$

*Proof.* Let  $\varphi \in C_c^\infty(\mathbb{R}^d)$  and  $\theta \in C^\infty(\mathbb{R})$  be such that

$$\int_{\mathbb{R}^d} \varphi = 1, \quad \varphi(x) = 0 \text{ for } |x| > 1,$$

$$\theta = 1 \text{ in } (-a/2, a/2), \quad \theta = 0 \text{ in } \mathbb{R} \setminus (-a, a), \quad \text{and} \quad |\theta'| \leq C_1/a \text{ in } \mathbb{R},$$

with  $a = \beta^{-1/2}$  for some positive constant  $C_1$  independent of  $a$ , and set, for  $t > 0$

$$\varphi_t(\cdot) \triangleq t^{-d} \varphi(\cdot/t) \quad \text{in } \mathbb{R}^d.$$



We define the function  $U : \mathbb{R}_+^{d+1} \rightarrow \mathbb{C}$  by setting, for every  $(x, t) \in \mathbb{R}^d \times [0, +\infty)$ ,

$$(3.1) \quad U(x, t) \triangleq \theta(t) \int_{\mathbb{R}^d} \varphi_t(x-y) e^{i\mathcal{I}_A((x,t),y)} u(y) dy,$$

where we have identified the point  $(y, 0) \in \mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$  with  $y \in \mathbb{R}^d$ .

By (2.5), for every  $(x, t) \in \mathbb{R}_+^{d+1}$ , we have, for  $1 \leq j \leq d$ ,

$$\partial_j U(x, t) + iA_j(x, t)U(x, t) = \theta(t) \int_{\mathbb{R}^d} \partial_j \varphi_t(x-y) e^{i\mathcal{I}_A((x,t),y)} u(y) dy$$

and

$$\begin{aligned} & \partial_{d+1} U(x, t) + iA_{d+1}(x, t)U(x, t) \\ &= \int_{\mathbb{R}^d} \left( \theta'(t)\varphi_t(x-y) - \theta(t) \left( \frac{d}{t} \varphi_t(x-y) - \frac{\theta(t)}{t^{d+2}} \nabla \varphi \left( \frac{x-y}{t} \right) \cdot (x-y) \right) \right) e^{i\mathcal{I}_A((x,t),y)} u(y) dy. \end{aligned}$$

With the notations  $\Phi_t^i(z) \triangleq \partial_j \varphi_t(z)$  if  $1 \leq j \leq d$  and  $\Phi_t^{d+1}(z) = -\frac{d}{t} \varphi_t(z) - t^{-(d+2)} \nabla \varphi(z/t) \cdot z$  for  $z \in \mathbb{R}^d$ , we have, since  $\int_{\mathbb{R}^d} \Phi_t^i = 0$ ,

$$(3.2) \quad \begin{aligned} & \int_{\mathbb{R}^d} \Phi_t^i(x-y) e^{i\mathcal{I}_A((x,t),y)} u(y) dy \\ &= \int_{\mathbb{R}^d} \Phi_t^i(x-y) \left( e^{i\mathcal{I}_A((x,t),y)} - e^{i\mathcal{I}_A(x,y) + i\mathcal{I}_A((x,t),x)} \right) u(y) dy \\ & \quad + \int_{\mathbb{R}^d} \Phi_t^i(x-y) e^{i\mathcal{I}_A((x,t),x)} \left( e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x) \right) dy. \end{aligned}$$

It follows that, for every  $(x, t) \in \mathbb{R}_+^{d+1}$ ,

$$(3.3) \quad |\nabla_A U(x, t)| \leq C_2 (L_1(x, t) + L_2(x, t) + L_3(x, t)),$$

where the functions  $L_1, L_2, L_3 : \mathbb{R}_+^{d+1} \rightarrow \mathbb{R}$  are defined for each  $(x, t) \in \mathbb{R}_+^{d+1}$  by

$$\begin{aligned} L_1(x, t) &\triangleq \frac{\mathbb{1}_{(0,a)}(t)}{t^{d+1}} \int_{B(x,t)} \left| e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x) \right| dy, \\ L_2(x, t) &\triangleq \frac{\mathbb{1}_{(0,a)}(t)}{t^{d+1}} \int_{B(x,t)} \left| e^{i(\mathcal{I}_A((x,t),y) + \mathcal{I}_{A''}(y,x) + \mathcal{I}_A(x,(x,t)))} - 1 \right| |u(y)| dy, \\ L_3(x, t) &\triangleq \frac{\mathbb{1}_{(0,a)}(t)}{a t^d} \int_{B(x,t)} |u(y)| dy. \end{aligned}$$

We first have, by Hölder's inequality,

$$(3.4) \quad \begin{aligned} & \iint_{\mathbb{R}_+^{d+1}} \frac{|L_1(x, t)|^p}{t^{1-(1-s)p}} dx dt \leq C_3 \iint_{\mathbb{R}_+^{d+1}} \frac{1}{t^{1+d+sp}} \left( \int_{B(x,t)} |e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x)|^p dy \right) dx dt \\ & \leq C_4 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy. \end{aligned}$$

Next, by Lemma 2.2, if  $x, y \in \mathbb{R}^d$  and  $t \in (0, +\infty)$ , we have

$$|\mathcal{I}_A((x, t), y) + \mathcal{I}_A(y, x) + \mathcal{I}_A(x, (x, t))| \leq C_5 \|dA\|_{L^\infty(\mathbb{R}_+^{d+1})} t^2,$$

and therefore,  $|y-x| \leq t$ ,

$$\left| e^{i(\mathcal{I}_A((x,t),y) + \mathcal{I}_A(y,x) + \mathcal{I}_A(x,(x,t)))} - 1 \right| \leq C_6 \beta^{1/2} t.$$

It follows from Fubini's theorem that

$$(3.5) \quad \begin{aligned} \iint_{\mathbb{R}_+^{d+1}} \frac{|L_2(x,t)|^p}{t^{1-(1-s)p}} dx dt &\leq C_7 \iint_{\mathbb{R}^d \times (0,a)} \frac{\beta^{p/2}}{t^{d+1-(1-s)p}} \int_{B(x,t)} |u(y)|^p dy dx dt \\ &= C_8 \int_0^a \frac{\beta^{\frac{p}{2}}}{t^{1-(1-s)p}} \int_{\mathbb{R}^d} |u(y)|^p dy dt = C_9 \beta^{\frac{sp}{2}} \int_{\mathbb{R}^d} |u(y)|^p dy. \end{aligned}$$

Finally, we have

$$(3.6) \quad \iint_{\mathbb{R}_+^{d+1}} \frac{|L_3(x,t)|^p}{a^p t^{1-(1-s)p}} dx dt \leq C_{10} \int_{\mathbb{R}^d} \int_{(0,a)} |u(y)|^p dy dx dt \leq C_{11} \beta^{\frac{sp}{2}} \int_{\mathbb{R}^d} |u|^p.$$

Combining (3.3), (3.4), (3.5), and (3.6) yields

$$(3.7) \quad \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p}{t^{1-(1-s)p}} dx dt \leq C_{12} \|u\|_{W_{A''}^{s,p}(\mathbb{R}^d)}^p + C_{13} \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p.$$

Similar to (3.6), we also have

$$(3.8) \quad \iint_{\mathbb{R}_+^{d+1}} \frac{|U(x,t)|^p}{t^{1-(1-s)p}} dx dt \leq C_{14} \frac{\|u\|_{L^p(\mathbb{R}^d)}^p}{\beta^{\frac{(1-s)p}{2}}}.$$

The conclusion now follows from (3.7) and (3.8).  $\square$

*Remark 3.2.* If in the proof of Proposition 3.1, one takes  $a = 1$ , then the following estimate holds:

$$\iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p + |U(x,t)|^p}{t^{1-(1-s)p}} dx dt \leq C \left( \|u\|_{W_{A''}^{s,p}(\mathbb{R}^d)}^p + (1 + \|dA\|_{L^\infty}^{\frac{p}{2}}) \|u\|_{L^p(\mathbb{R}^d)}^p \right).$$

We also have a local version of Proposition 3.1.

**Proposition 3.3.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . There exists a constant  $C_{d,s,p}$  such that for every  $u \in C^\infty(B(0,2R), \mathbb{C})$  and every  $A \in C^1(\bar{B}_{2R} \times [0,R], \wedge^1 \mathbb{R}^{d+1})$  such that  $\|dA\|_{L^\infty(B(0,2R) \times [0,R])} + \frac{1}{R^2} \leq \beta$ , there exists  $U \in C^\infty(B(0,R) \times (0,R), \mathbb{C})$  such that  $U(\cdot, 0) = u(\cdot)$  on  $B(0,R)$  and*

$$\begin{aligned} \iint_{B(0,R) \times (0,R)} \frac{|\nabla_A U(x,t)|^p}{t^{1-(1-s)p}} dx dt \\ \leq C_{d,s,p} \left( \iint_{B(0,2R) \times B(0,2R)} \frac{|e^{i\mathcal{I}_{A''}(x,y)} u(y) - u(x)|^p}{|x-y|^{d+sp}} dx dy + \beta^{\frac{sp}{2}} \int_{B(0,2R)} |u|^p \right) \end{aligned}$$

and

$$\int_{B(0,R) \times (0,R)} \frac{|U(z,t)|^p}{t^{1-(1-s)p}} dt dz \leq \frac{C_{d,s,p}}{\beta^{\frac{(1-s)p}{2}}} \int_{B(0,2R)} |u|^p.$$

#### 4. CHARACTERIZATIONS OF TRACE SPACES

For  $\gamma \in \mathbb{R}$ , we define the weighted space

$$(4.1) \quad W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1}) \triangleq \left\{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+^{d+1}, \mathbb{C}); \|U\|_{W_{A,\gamma}^{1,p}(\Omega)} < +\infty \right\},$$

where

$$\|U\|_{W_{A,\gamma}^{1,p}(\Omega)}^p \triangleq \iint_{\mathbb{R}_+^{d+1}} \left( |U(x,t)|^p + |\nabla_A U(x,t)|^p \right) t^\gamma dx dt$$

It is standard to check that the space  $W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$  is complete. We also have the following density result:

**Lemma 4.1.** *Let  $1 \leq p < +\infty$ ,  $\gamma \in \mathbb{R}$  and  $A \in C(\mathbb{R}_+^{d+1}, \mathbb{R}^{d+1})$ . If  $1 - p < \gamma < 1$ , then the space  $C_c^\infty(\overline{\mathbb{R}_+^{d+1}})$  is dense in  $W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$ .*

*Proof.* The proof of the completeness is standard. For the density of smooth maps, we first observe that if  $\chi \in C_c^\infty(\mathbb{R}^{d+1})$ ,  $0 \leq \chi \leq 1$ , and  $\chi = 1$  on  $B(0,1)$  and if we define for each  $\lambda > 0$  the function  $\chi_\lambda : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  for  $x \in \mathbb{R}^d$  by  $\chi_\lambda(x) = \chi(x/\lambda)$ , then  $\nabla_A(U - \chi_\lambda U) = -U\nabla\chi_\lambda + (1 - \chi)\nabla_A U$ . It follows that

$$\begin{aligned} & \|U - \chi_\lambda U\|_{W_{A,\gamma}^{1,p}(\mathbb{R}^d)}^p \\ & \leq C_1 \iint_{\mathbb{R}_+^{d+1}} \left( (1 - \chi_\lambda(x,t))^p (|U(x,t)|^p + |\nabla_A U(x,t)|^p) + |\nabla\chi_\lambda|^p |U(x,t)|^p \right) t^\gamma dx dt \rightarrow 0, \end{aligned}$$

as  $\lambda \rightarrow \infty$ , since  $|\nabla\chi_\lambda| \leq C_2/\lambda$ . Functions in  $W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$  with bounded support are thus dense in  $W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$ .

Since  $1 - p < \gamma < 1$ , any compactly supported can be approximated in  $W_{0,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$  by smooth functions with bounded support [21, Lemma 2.4; 24, Lemma 8; 33, Corollary 2.1.5] (the condition ensures that the weight  $(x,t) \mapsto t^\gamma$  satisfies Muckenhoupt's  $A_p$  condition for  $p \in [1, +\infty)$  given by the assumptions) [7, 23]. Since the function  $A$  is locally bounded on  $\mathbb{R}^{d+1}$ , this implies that any bounded supported function in  $W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$  can be approximated in  $W_{0,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$  by smooth functions with bounded support and the conclusion follows by a diagonal argument.  $\square$

It is also standard to check that the space  $W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)$  is complete and thus is a Banach space. We also have the following density result:

**Lemma 4.2.** *Let  $0 < s < 1$  and  $p \geq 1$ . If  $A^{\text{II}} \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ , then the space  $C_c^\infty(\mathbb{R}^d)$  is dense in  $W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)$ .*

*Proof.* First we observe that if  $u \in W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)$  is arbitrary,  $\chi \in C_c^\infty(\mathbb{R}^d)$  is chosen with  $0 \leq \chi \leq 1$  and  $\chi = 1$  for  $x \in B(0,1)$ , and  $\chi_\lambda(x) \triangleq \chi(x/\lambda)$ , then we have for each  $x, y \in \mathbb{R}^d$  and  $\lambda > 0$ ,

$$\begin{aligned} & (1 - \chi_\lambda(y)) e^{i\mathcal{I}_{A^{\text{II}}}(x,y)} u(y) - (1 - \chi_\lambda(x)) u(x) \\ & = \left(1 - \frac{\chi_\lambda(x) + \chi_\lambda(y)}{2}\right) (e^{i\mathcal{I}_{A^{\text{II}}}(x,y)} u(y) - u(x)) + \frac{\chi_\lambda(x) - \chi_\lambda(y)}{2} (e^{i\mathcal{I}_{A^{\text{II}}}(x,y)} u(y) + u(x)), \end{aligned}$$

and for every  $y \in \mathbb{R}^d$  and  $\lambda > 0$ ,

$$\int_{\mathbb{R}^d} \frac{|\chi_\lambda(x) - \chi_\lambda(y)|^p}{|y - x|^{d+sp}} dx \leq \frac{C_1}{\lambda^{sp}}.$$

It follows that, for every  $\lambda > 0$ ,

$$\begin{aligned} & |\chi_\lambda u - u|_{W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)}^p \\ & \leq C_2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(1 - \frac{\chi_\lambda(x) + \chi_\lambda(y)}{2}\right)^p \frac{|e^{i\mathcal{I}_{A^{\text{II}}}(x,y)} u(y) - u(x)|^p}{|y - x|^{d+sp}} dy dx + C_3 \int_{\mathbb{R}^d} \frac{|u(y)|^p}{\lambda^{sp}} dy, \end{aligned}$$

By Lebesgue's dominated convergence theorem we deduce that

$$|\chi_\lambda u - u|_{W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)} \rightarrow 0 \quad \text{and} \quad \|\chi_\lambda u - u\|_{L^p(\mathbb{R}^d)} \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Hence functions with compact support are dense in  $W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)$ .

We conclude by observing that since  $A^{\text{II}}$  is locally bounded, any function in  $W_{A^{\text{II}}}^{s,p}(\mathbb{R}^d)$  with compact support is in  $W_0^{s,p}(\mathbb{R}^d)$ , such functions can be approximated in  $W_0^{s,p}(\mathbb{R}^d)$  by functions

with uniformly compact support; since the function  $A^\flat$  is locally bounded, the approximating sequence also converges in  $W_{A^\flat}^{s,p}(\mathbb{R}^d)$ . The conclusion then follows by a diagonal argument.  $\square$

As a consequence of Propositions 2.1, 3.1, 2.1, and 3.1, and Lemmas 4.1 and 4.2, we obtain

**Theorem 4.3.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . If  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$  and  $\|dA\|_{L^\infty(\mathbb{R}_+^{d+1})} \leq \beta$ , then there exists a trace mapping  $\text{Tr} : W_{A,1-(1-s)p}^{1,p}(\mathbb{R}_+^{d+1}, \mathbb{C}) \rightarrow W_{A^\flat}^{s,p}(\mathbb{R}^d)$  such that for some positive constant  $C_{d,s,p}$  depending only on  $d$ ,  $s$  and  $p$ , if  $U \in C_c^1(\mathbb{R}_+^{d+1}, \mathbb{C})$ , then  $\text{Tr} U = U(\cdot, 0)$  and for every  $U \in W_{A,1-(1-s)p}^{1,p}(\mathbb{R}_+^{d+1})$ , if  $u \triangleq \text{Tr} U$ ,*

$$|u|_{W_{A^\flat}^{s,p}(\mathbb{R}^d)}^p + \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p \leq C_{d,s,p} \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p + \beta^{\frac{p}{2}} |U(z,t)|^p}{t^{1-(1-s)p}} dx dt,$$

and there exists a linear continuous mapping  $\text{Ext} : W_{A^\flat}^{s,p}(\mathbb{R}^d, \mathbb{C}) \rightarrow W_{A,1-(1-s)p}^{1,p}(\mathbb{R}_+^{d+1})$  such that  $\text{Tr} \circ \text{Ext} : W_{A^\flat}^{s,p}(\mathbb{R}^d) \rightarrow W_{A^\flat}^{s,p}(\mathbb{R}^d)$  is the identity and such that for some positive constant  $C_{d,s,p}$  depending only on  $d$ ,  $s$  and  $p$ , we have for each  $u \in W_{A^\flat}^{s,p}(\mathbb{R}^d)$ , if  $U \triangleq \text{Ext} u$ ,

$$\iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p + \beta^{\frac{p}{2}} |U(z,t)|^p}{t^{1-(1-s)p}} dx dt \leq C_{d,s,p} \left( |u|_{W_{A^\flat}^{s,p}(\mathbb{R}^d)}^p + \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p \right).$$

## 5. CONSTANT MAGNETIC FIELD ON THE HALF-SPACE

We begin with an improvement of Proposition 2.1 in the case where the magnetic field  $dA$  is constant.

**Proposition 5.1.** *Let  $d \geq 1$ ,  $0 < s < 1$ , and  $1 \leq p < +\infty$ . There exists a constant  $C_{d,s,p} > 0$  such that if  $A \in C^1(\mathbb{R}_+^{d+1}, \mathbb{R}^{d+1})$  with constant  $dA$ ,  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}})$  and  $u = U(\cdot, 0)$ , then*

$$|u|_{W_{A^\flat}^{s,p}(\mathbb{R}^d)}^p + \|dA\|^{\frac{sp}{2}} \int_{\mathbb{R}^d} |U(x,0)|^p dx \leq C_{d,s,p} \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z,t)|^p}{t^{1+(1-s)p}} dz dt.$$

The first ingredient of the proof Proposition 5.1 is the following lemma:

**Lemma 5.2.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . For every  $\lambda > 0$ , there exists a positive constant  $C_{d,s,p,\lambda}$  such that, for every  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \wedge^1 \mathbb{R}^{d+1})$  with constant  $dA$ , we have, for  $U \in C_c^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{C})$ ,*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i(\mathcal{I}_{A^\flat}(x,y) + \lambda|y-x|dA[y-x, e_{d+1}])} U(y,0) - U(x,0)|^p}{|y-x|^{d+sp}} dx dy \leq C_{d,s,p,\lambda} \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z,t)|^p}{t^{1-(1-s)p}} dz dt.$$

*Proof.* For  $x, y \in \mathbb{R}^d$ , we identify  $x = (x, 0)$ ,  $y = (y, 0)$ , and we set  $Z = (\frac{x+y}{2}, 2\lambda|y-x|)$ . Since  $dA$  is constant, by Lemma 2.2, we have

$$\mathcal{I}_A(x, y) + \mathcal{I}_A(y, Z) + \mathcal{I}_A(Z, x) = \frac{1}{2} dA[(y-x, 0), (\frac{x-y}{2}, -2\lambda|x-y|)] = -\lambda \nu_A(y-x),$$

where the function  $\nu_A : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined for each  $z \in \mathbb{R}^d$  by  $\nu_A(z) = |z| dA[z, e_{d+1}]$ . This implies by the triangle inequality

$$(5.1) \quad \left| e^{i(\mathcal{I}_A(x,y) + \lambda \nu_A(y-x))} U(y) - U(x) \right| \leq \left| e^{i\mathcal{I}_A(x,Z)} U(Z) - U(x) \right| + \left| e^{i\mathcal{I}_A(y,Z)} U(y) - U(Z) \right|.$$

Using (2.6), we deduce from (5.1) that

$$\begin{aligned} & \left| e^{i(\mathcal{I}_A(x,y)+\lambda\nu_A(y-x))}U(y) - U(x) \right| \\ & \leq \sqrt{1+\lambda^2}|y-x| \left( \int_0^1 |\nabla_A U((1-\frac{t}{2})x + \frac{t}{2}y, 2\lambda|y-x|)| dt \right. \\ & \quad \left. + \int_0^1 |\nabla_A U((1-\frac{t}{2})y + \frac{t}{2}x, 2\lambda|y-x|)| dt \right). \end{aligned}$$

We then have, by Minkowski's inequality,

$$\begin{aligned} (5.2) \quad & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i(\mathcal{I}_A(x,y)+\lambda\nu_A(y-x))}U(y) - U(x)|^p}{|y-x|^{d+sp}} dx dy \\ & \leq C_1(1+\lambda)^p \left( \int_0^1 \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U((1-\frac{t}{2})x + \frac{t}{2}y, 2\lambda|y-x|)|^p}{|y-x|^{d-(1-s)p}} dy dx \right)^{1/p} \right. \\ & \quad \left. + \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U((1-\frac{t}{2})y + \frac{t}{2}x, 2\lambda|y-x|)|^p}{|y-x|^{d-(1-s)p}} dy dx \right)^{1/p} dt \right)^p. \end{aligned}$$

Performing the change of variable  $z = (1-\frac{t}{2})x + \frac{t}{2}y$  and  $v = 2\lambda t(x-y)$ , we obtain

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U((1-\frac{t}{2})x + \frac{t}{2}y, 2\lambda|y-x|)|^p}{|y-x|^{d-(1-s)p}} dy dx = \frac{1}{(2\lambda t)^{(1-s)p}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U(z, |v|)|^p}{|v|^{d-(1-s)p}} dz dv.$$

This yields by using spherical coordinates, for every  $t \in (0, 1)$ ,

$$(5.3) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U((1-\frac{t}{2})x + \frac{t}{2}y, 2\lambda|y-x|)|^p}{|y-x|^{d-(1-s)p}} dx dy \leq \frac{C_2}{(\lambda t)^{p(1-s)}} \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|\nabla_A U(z, r)|^p}{r^{1-(1-s)p}} dz dr.$$

Similarly, we have

$$(5.4) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla_A U((1-\frac{t}{2})y + \frac{t}{2}x, 2\lambda|y-x|)|^p}{|y-x|^{d-(1-s)p}} dx dy \leq \frac{C_2}{(\lambda t)^{p(1-s)}} \iint_{\mathbb{R}^d \times (0, +\infty)} \frac{|\nabla_A U(z, r)|^p}{r^{1-(1-s)p}} dz dr.$$

Combining (5.2) with (5.3), and (5.4) yields

$$\begin{aligned} (5.5) \quad & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i(\mathcal{I}_{A^u}(x,y)+\lambda\nu_A(y-x))}U(y, 0) - U(x, 0)|^p}{|y-x|^{d+sp}} dx dy \\ & \leq \frac{C_3(1+\lambda)^p}{\lambda^p} \left( \int_0^1 \frac{1}{t^{1-s}} dt \right)^p \int_{\mathbb{R}_+^{d+1}} |\nabla_A U(z, r)|^p dz dr \end{aligned}$$

and the conclusion follows.  $\square$

The second tool is the following fractional magnetic Poincaré inequality.

**Lemma 5.3.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$  and  $A^u \in C^1(\mathbb{R}^d, \wedge^1 \mathbb{R}^d)$  with constant  $dA$ . There exists a constant  $C_{d,s,p} > 0$  such that, for  $u \in C_c^1(\mathbb{R}^d)$ ,*

$$\int_{\mathbb{R}^d} |u|^p \leq \frac{C_{d,s,p}}{\|dA^u\|_{\frac{sp}{2}}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A^u}(x,y)}u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy.$$

The counterpart of Lemma 5.3 in  $W_A^{1,2}(\mathbb{R}^d)$  is known [4, Theorem 2.9; 11, Proposition 2.2] and related to the positiveness of the first eigenvalue of the magnetic Laplacian  $-\Delta_A$ , which corresponds to the first Landau level.

*Proof of Lemma 5.3.* Let  $J : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a linear isometry. We observe that by setting  $z = x + h$  and  $k = Jh$ , we have

$$(5.6) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(x+h, x+h+Jh)}u(x+h+Jh) - u(x+h)|^p}{|h|^{d+sp}} dx dh = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(z, z+k)}u(z+k) - u(z)|^p}{|k|^{d+sp}} dz dk.$$

Similarly, by setting  $z = x + h + Jh$  and  $k = -h$ , we have

$$(5.7) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(x+h+Jh, x+Jh)}u(x+Jh) - u(x+h+Jh)|^p}{|h|^{d+sp}} dx dh = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(z, z+k)}u(z+k) - u(z)|^p}{|k|^{d+sp}} dz dk.$$

Finally, we have by setting  $z = x + Jh$  and  $k = -h$ ,

$$(5.8) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(x+Jh, x)}u(x) - u(x+Jh)|^p}{|h|^{d+sp}} dx dh = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(z, z+k)}u(z+k) - u(z)|^p}{|k|^{d+sp}} dz dk.$$

We compute now, since  $dA$  is constant, by Stokes formula

$$\mathcal{I}_A(x, x+h) + \mathcal{I}_A(x+h, x+h+Jh) + \mathcal{I}_A(x+Jh, x+h+Jh) + \mathcal{I}_A(x+Jh, h) = dA[h, Jh],$$

so that, by the triangle inequality,

$$(5.9) \quad |e^{dA[h, Jh]} - 1||u(x)| \leq |e^{i\mathcal{I}_A(x, x+h)}u(x+h) - u(x)| + |e^{i\mathcal{I}_A(x+h, x+h+Jh)}u(x+h+Jh) - u(x+h)| + |e^{i\mathcal{I}_A(x+Jh, x+h+Jh)}u(x+h) - u(x+h+Jh)| + |e^{i\mathcal{I}_A(x+Jh, h)}u(x) - u(x+Jh)|.$$

Therefore, we have by Hölder's inequality, in view of (5.6), (5.7) and (5.8),

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{dA[h, Jh]} - 1|^p |u(x)|^p}{|h|^{d+sp}} dx dh \leq 4^{p-1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(x, x+h)}u(x) - u(x+h)|^p}{|h|^{d+sp}} dx dh.$$

Integrating with respect to  $J$  over the group  $SO_d$  of rotations of  $\mathbb{R}^d$ , we obtain

$$\|dA\|^{sp/2} \int_{\mathbb{R}^d} |u(x)|^p dx \leq C_1 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_A(x, x+h)}u(x) - u(x+h)|^p}{|h|^{d+sp}} dx dh.$$

The proof is complete.  $\square$

We are ready to give

*Proof of Proposition 5.1.* Applying Lemma 5.2 with  $\lambda = 1$  and  $\lambda = 2$ , and using the triangle inequality, we obtain

$$(5.10) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i|y-x|dA[y-x, e_{d+1}]} - 1|^p |U(y, 0)|^p}{|y-x|^{d+sp}} dx dy \leq C_2 \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z, t)|^p}{t^{1-(1-s)p}} dz dt.$$

Since

$$\begin{aligned} |e^{i\mathcal{I}_{A''}(x,y)}U(y,0) - U(x,0)|^p &\leq 2^{p-1}|e^{i(\mathcal{I}_{A''}(x,y)+|y-x|dA[y-x,e_{d+1}])}U(y,0) - U(x,0)|^p \\ &\quad + 2^{p-1}|e^{i|y-x|dA[y-x,e_{d+1}]} - 1|^p|U(y,0)|^p, \end{aligned}$$

it follows from (5.10) and Lemma 5.2 with  $\lambda = 1$  that

$$(5.11) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}(x,y)}U(y,0) - U(x,0)|^p}{|y-x|^{d+sp}} dx dy \leq C_3 \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z,t)|^p}{t^{1-(1-s)p}} dz dt.$$

From Lemma 5.3, we have

$$(5.12) \quad \|dA''\|_{L^p(\mathbb{R}^d)}^{\frac{sp}{2}} \int_{\mathbb{R}^d} |U(\cdot,0)| \leq C_4 \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(z,t)|^p}{t^{1-(1-s)p}} dz dt.$$

Combining (5.11) and (5.12) yields the conclusion.  $\square$

Using Proposition 5.1 and Theorem 4.3, we obtain the following result which implies Theorem 1.3 in the introduction.

**Theorem 5.4.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . Assume that  $A \in C^1(\overline{\mathbb{R}_+^{d+1}}, \mathbb{R}^{d+1})$  and  $dA$  is constant. Then, with  $u \triangleq \text{Tr } U$ ,*

$$|u|_{W_{A''}^{s,p}(\mathbb{R}^d)}^p + \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p \leq C_{d,s,p} \iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p + \|dA\|_{L^p}^{\frac{p}{2}} |U(z,t)|^p}{t^{1-(1-s)p}} dx dt,$$

and with  $U \triangleq \text{Ext } u$ ,

$$\iint_{\mathbb{R}_+^{d+1}} \frac{|\nabla_A U(x,t)|^p + \|dA\|_{L^p}^{\frac{p}{2}} |U(z,t)|^p}{t^{1-(1-s)p}} dx dt \leq C_{d,s,p} \left( |u|_{W_{A''}^{s,p}(\mathbb{R}^d)}^p + \beta^{\frac{sp}{2}} \|u\|_{L^p(\mathbb{R}^d)}^p \right),$$

for some positive constant  $C_{d,s,p}$  depending only on  $d$ ,  $s$  and  $p$ ,

## 6. TRACE AND EXTENSION ON DOMAINS

In this section, we consider the trace problem on a domain  $\Omega$  of class  $C^1$  with estimates depending only on the magnetic field  $dA$ . We first develop the tools to work with a magnetic derivative on the boundary  $\partial\Omega$  via local charts. Let  $W \subset \mathbb{R}^d$  be an open set,  $\psi \in C^1(W, \mathbb{R}^d)$ , and  $A : \psi(W) \rightarrow \wedge^1 \mathbb{R}^d$ . The pull-back  $\psi^*A$  of  $A$  by  $\psi$  is defined for each  $x \in W$  and  $v \in \mathbb{R}^n$  by

$$(6.1) \quad (\psi^*A)(x) \triangleq D\psi(x)^* A(\psi(x)),$$

where  $D\psi(x)^*$  is the adjoint of  $D\psi(x)$ . We first recall the following elementary result whose proof follows from the chain rule (see, e.g., [36, Proposition 6.1.11]) and the definition of the pull-back (6.1).

**Lemma 6.1.** *Let  $d \geq 1$ ,  $V \subset \mathbb{R}^d$  be open, bounded, and let  $\psi \in C^1(V, \mathbb{R}^{d+1})$  be a diffeomorphism on its image up to the boundary. Then  $U \in W_A^{1,p}(\psi(V))$  if and only if  $U \circ \psi \in W_{\psi^*A}^{1,p}(V)$ . Moreover, for almost every  $x \in V$ ,*

$$(6.2) \quad \nabla_{\psi^*A}(U \circ \psi)(x) = D\psi(x)^*[(\nabla_A U)(\psi(x))].$$

Consequently,

$$(6.3) \quad \frac{|\nabla_A U(\psi(x))|}{\|D\psi^{-1}\|_{L^\infty}} \leq |\nabla_{\psi^*A}(U \circ \psi)(x)| \leq \|D\psi\|_{L^\infty} |\nabla_A U(\psi(x))|.$$

We next derive a similar result on the boundary when  $\psi : W \mapsto \partial\Omega$  for the fractional magnetic Gagliardo–Sobolev seminorm. Although the potential  $\mathcal{I}_{\bar{A}}$  defined in (2.4) does not make sense in general for each  $x, y \in \partial\Omega$  if  $\Omega$  is not convex for  $A^i \in C^1(\wedge^1 T^* \partial\Omega)$ , when  $\partial\Omega$  is compact smooth manifold, then  $\partial\Omega$  has a positive injectivity radius  $\text{inj}_{\partial\Omega}$  and if  $x, y \in \partial\Omega$  and  $d_{\partial\Omega}(y, x) \leq \text{inj}_{\partial\Omega}$ , then there exists a unique minimizing geodesic  $\gamma : [0, 1] \rightarrow \partial\Omega$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . We then define for such  $x$  and  $y$  the quantity

$$(6.4) \quad \mathcal{I}_{A^i}^{\partial\Omega}(x, y) \triangleq \int_0^1 A^i(\gamma(t)) \cdot \gamma'(t) dt.$$

We have

**Lemma 6.2.** *Let  $d \geq 1$ ,  $W \subset \mathbb{R}^d$  be open, bounded, and let  $\psi : W \rightarrow \mathbb{R}^{d+1}$  be a diffeomorphism up to the boundary to its image as a subset of the manifold  $\partial\Omega$ . If  $W$  and  $\psi(W)$  are geodesically convex, then for every  $0 < s < 1$  and  $1 \leq p < +\infty$ , there exists a positive constant  $C$  such that for every  $A^i \in C^1(\wedge^1 T^* \partial\Omega)$  and every measurable function  $u : \psi(W) \rightarrow \mathbb{C}$ ,*

$$\begin{aligned} & C^{-1} \iint_{\psi(W) \times \psi(W)} \frac{|e^{i\mathcal{I}_{A^i}^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{d_{\partial\Omega}(y, x)^{d+sp}} dy dx \\ & \leq \iint_{W \times W} \frac{|e^{i\mathcal{I}_{\psi^* A^i}(x,y)} u(\psi(y)) - u(\psi(x))|^p}{|y - x|^{d+sp}} dy dx + \min(\|dA^i\|_{L^\infty}^p, \|dA^i\|_{L^\infty}^{\frac{sp}{3}}) \int_W |u \circ \psi|^p dx \end{aligned}$$

and

$$\begin{aligned} & \iint_{W \times W} \frac{|e^{i\mathcal{I}_{\psi^* A^i}(x,y)} u(\psi(y)) - u(\psi(x))|^p}{|y - x|^{d+sp}} dy dx \\ & \leq C \left( \iint_{\psi(W) \times \psi(W)} \frac{|e^{i\mathcal{I}_{A^i}^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{d_{\partial\Omega}(y, x)^{d+sp}} dy dx + \min\{\|dA^i\|_{L^\infty}^p, \|dA^i\|_{L^\infty}^{\frac{sp}{3}}\} \int_{\psi(W)} |u|^p dx \right). \end{aligned}$$

In the proof of Lemma 6.2, we use the following result which relates the potential  $\mathcal{I}_{\bar{A}}$  to the potential  $\mathcal{I}_{\psi^* A}$  via local charts.

**Lemma 6.3.** *Let  $d \geq 1$ , let  $W \subset \mathbb{R}^d$  be open and bounded, and let  $\psi : W \rightarrow \partial\Omega$  be a diffeomorphism up to the boundary to its image. If  $W$  and  $\psi(W)$  are geodesically convex, then there exists a positive constant  $C$  such that for every  $x, y \in W$ , we have*

$$|\mathcal{I}_{\psi^* A^i}(x, y) - \mathcal{I}_{A^i}^{\partial\Omega}(\psi(x), \psi(y))| \leq C \|dA^i\|_{L^\infty(\psi(W))} |y - x|^3.$$

*Proof.* For every  $x, y \in W$ , there exists a unique minimizing geodesic  $\gamma : [0, 1] \rightarrow \psi(W)$  such that  $\gamma(0) = \psi(x)$  and  $\gamma(1) = \psi(y)$ . Since  $\gamma$  is a geodesic, the function  $\tilde{\gamma} = \psi^{-1} \circ \gamma$  satisfies the equation

$$\tilde{\gamma}''(t) = \Gamma(\tilde{\gamma}(t))[\tilde{\gamma}'(t), \tilde{\gamma}'(t)],$$

where for every  $z \in \psi(W)$ ,  $\Gamma(z)$  is a symmetric bilinear mapping (see, e.g., [10, Chapter 3]). There exists thus a constant  $C_1$  such that for every  $t \in [0, 1]$ ,

$$|\tilde{\gamma}''(t)| \leq C_1 |y - x|^2.$$

Since  $\tilde{\gamma}(0) = x$  and  $\tilde{\gamma}(1) = y$ , we deduce that for every  $t \in [0, 1]$  we have

$$|(1-t)x + ty - \tilde{\gamma}(t)| \leq C_2 t(1-t) |y - x|^2.$$

We have then by the Stokes theorem

$$\begin{aligned} & \mathcal{I}_{\psi^* A^i}(y, x) - \mathcal{I}_{A^i}^{\partial\Omega}(\psi(y), \psi(x)) \\ & = \int_{[0,1]^2} dA((1-s)((1-t)x + ty) + s\tilde{\gamma}(t))[(1-t)x + ty - \tilde{\gamma}(t), (1-s)\tilde{\gamma}'(t) + s(y-x)] dt ds, \end{aligned}$$



and therefore,

$$|\mathcal{I}_{\psi^*A^{\text{II}}}(\psi(y), \psi(x)) - \mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(\psi(y), \psi(x))| \leq C_3 \|dA^{\text{II}}\|_{L^\infty(\psi(W))} |y - x|^3. \quad \square$$

We are ready to give the

*Proof of Lemma 6.2.* By the change of variable formula, we have

$$\begin{aligned} & \iint_{\psi(W) \times \psi(W)} \frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{d_{\partial\Omega}(y,x)^{d+sp}} dy dx \\ &= \iint_{W \times W} \frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(\psi(x), \psi(y))} u(\psi(y)) - u(\psi(x))|^p}{d_{\partial\Omega}(\psi(y), \psi(x))^{d+sp}} \text{Jac } \psi(y) \text{Jac } \psi(x) dy dx. \end{aligned}$$

We thus obtain

$$\begin{aligned} & \iint_{\psi(W) \times \psi(W)} \frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{d_{\partial\Omega}(y,x)^{d+sp}} dy dx \\ & \leq C_1 \left( \iint_{W \times W} \frac{|e^{i\mathcal{I}_{\psi^*A^{\text{II}}}(x,y)} u(\psi(y)) - u(\psi(x))|^p}{|y - x|^{d+sp}} dy dx \right. \\ & \quad \left. + \iint_{W \times W} \frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(\psi(x), \psi(y))} - e^{i\mathcal{I}_{\psi^*A^{\text{II}}}(x,y)}|^p |u(\psi(y))|^p}{|y - x|^{d+sp}} dy dx \right). \end{aligned}$$

Since, by Lemma 6.3, for every  $x, y \in W$ ,

$$\frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(\psi(x), \psi(y))} - e^{i\mathcal{I}_{\psi^*A^{\text{II}}}(x,y)}|^p}{|y - x|^{d+sp}} \leq \frac{C_2 \min\{\|dA^{\text{II}}\|_{L^\infty(\psi(W))}^p |y - x|^{3p}, 1\}}{|y - x|^{d+sp}},$$

the first estimate then follows from the facts

$$\int_{\mathbb{R}^d} \frac{\min\{\|dA^{\text{II}}\|_{L^\infty(\psi(W))}^p |z|^{3p}, 1\}}{|z|^{d+sp}} dz \leq C_3 \|dA^{\text{II}}\|_{L^\infty(\psi(W))}^{\frac{sp}{3}}$$

and

$$\int_{B(0, \text{diam}(W))} \frac{\|dA^{\text{II}}\|_{L^\infty(\psi(W))}^p |z|^{3p}}{|z|^{d+sp}} dz \leq C_4 \|dA^{\text{II}}\|_{L^\infty(\psi(W))}^p.$$

The proof of the second estimate follows similarly.  $\square$

We have the following estimate on the traces of magnetic Sobolev spaces.

**Proposition 6.4.** *Let  $d \geq 1$ , let  $1 \leq p < +\infty$  and let  $\Omega \subset \mathbb{R}^{d+1}$  be a bounded domain of class  $C^1$ . There exists a constant  $C_{\Omega,p} > 0$  such that if  $A \in C^1(\bar{\Omega}, \wedge^1 \mathbb{R}^d)$  and  $\|dA\|_{L^\infty(\Omega)} \leq \beta$  and  $U \in W_A^{1,p}(\Omega)$ , then  $u \triangleq \text{Tr } U$  satisfies the estimate*

$$\iint_{\substack{(x,y) \in \partial\Omega \times \partial\Omega \\ d_{\partial\Omega}(y,x) \leq \text{inj}_{\partial\Omega}}} \frac{|e^{i\mathcal{I}_{A^{\text{II}}}^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{|y - x|^{d+p-1}} dx dy \leq C_{\Omega,p} \int_{\Omega} |\nabla_A U|^p + (1 + \beta^{\frac{p}{2}}) |U|^p.$$

Here for  $z \in \partial\Omega$ ,  $A^{\text{II}}(z) \triangleq A(z) - (A(z) \cdot \nu(z))\nu(z)$  where  $\nu(z)$  denotes a unit normal vector of  $\partial\Omega$  at  $z$ .

*Proof of Proposition 6.4.* Without loss of generality, we can assume that  $\beta \geq 1$ . By density argument on  $W^{1,p}(\Omega, \mathbb{C})$ , we can assume that  $u \in C^\infty(\bar{\Omega})$ . We first observe that, by the classical trace theory and the diamagnetic inequality,

$$(6.5) \quad \int_{\partial\Omega} |u|^p \leq C_1 \left( \int_{\Omega} |\nabla_A U|^p + |U|^p \right)^{\frac{1}{p}} \left( \int_{\Omega} |U|^p \right)^{1-1/p}.$$

Since  $\partial\Omega$  is compact and  $\Omega$  is of class  $C^1$ , there exists maps  $\psi_i : B(0,1) \times (-1,1) \rightarrow \mathbb{R}^d$  that are diffeomorphism on their image such that  $\psi_i(B(0,1) \cap (0,1)) = \psi_i(B(0,1) \times (0,1)) \cap \Omega$ ,  $\psi_i(B(0,1) \times \{0\}) = \psi_i(B(0,1)) \cap \partial\Omega$ ,  $\partial\Omega \subset \bigcup_{i=1}^{\ell} \psi_i(B(0,1/2) \times \{0\})$  and for every  $i \in \{1, \dots, \ell\}$ ,  $\psi_i(B(0,1) \times \{0\})$  is geodesically convex. Applying Proposition 2.4 for  $i \in \{1, \dots, \ell\}$ , we have

$$\iint_{B(0,1) \times B(0,1)} \frac{|e^{i\mathcal{I}_{\psi_i^* A^u}(x,y)}(u(\psi_i(y,0)) - u(\psi_i(x,0)))|^p}{|y-x|^{d+p-1}} dx dy \leq C_2 \int_{B(0,1) \times [0,1]} |\nabla_A(U \circ \psi_i)|^p + \beta^{\frac{p}{2}} |U \circ \psi_i|^p.$$

Since  $\beta \geq 1$ , in view of Lemma 6.1 and Lemma 6.2, this implies that

$$(6.6) \quad \iint_{\psi_i(B(0,1) \times \{0\}) \times \psi_i(B(0,1) \times \{0\})} \frac{|e^{i\mathcal{I}_{A^u}^{\partial\Omega}(x,y)}u(y) - u(x)|^p}{d_{\partial\Omega}(y,x)^{d+p-1}} dx dy \leq C_3 \left( \int_{\psi_i(B(0,1) \times (0,1))} |\nabla_A U|^p + \beta^{\frac{p}{2}} |U|^p + \beta^{\frac{p-1}{3}} \int_{\psi_i(B(0,1) \times \{0\})} |u|^p \right).$$

We have then by Young's inequality and by (6.5)

$$(6.7) \quad \beta^{\frac{p-1}{3}} \int_{\psi_i(B(0,1) \times \{0\})} |u|^p \leq \beta^{\frac{p-1}{2}} \int_{\psi_i(B(0,1) \times \{0\})} |u|^p \leq C_4 \int_{\psi_i(B(0,1) \times \{0\})} |\nabla_A U|^p + (1 + \beta^{\frac{p}{2}}) |U|^p.$$

The conclusion follows by summing for  $i \in \{1, \dots, \ell\}$  the estimates resulting the combination of (6.6) and (6.7).  $\square$

Concerning the extension, we have

**Proposition 6.5.** *Let  $d \geq 1$ , let  $1 \leq p < +\infty$  and let  $\Omega \subset \mathbb{R}^d$  be a bounded domain of class  $C^1$ . There exists a constant  $C_{\Omega,p} > 0$  such that if  $A \in C^1(\bar{\Omega}, \wedge^1 \mathbb{R}^d)$ , if  $\|dA\|_{L^\infty(\Omega)} \leq \beta$  and if  $u \in W^{1-1/p,p}(\partial\Omega)$ , then there exists  $U \in W^{1,p}(\Omega) \cap C^\infty(\Omega, \mathbb{C})$  such that  $\text{Tr}_{\partial\Omega} U = u$  and*

$$\int_{\Omega} |\nabla_A U|^p \leq C_{\Omega,p} \left( \iint_{\substack{(x,y) \in \Omega \times \Omega \\ d_{\partial\Omega}(y,x) \leq \text{inj}_{\partial\Omega}}} \frac{|e^{i\mathcal{I}_{A^u}^{\partial\Omega}(x,y)}u(y) - u(x)|^p}{|y-x|^{d+p-1}} dx dy + (1 + \beta^{\frac{p-1}{2}}) \int_{\partial\Omega} |u|^p \right)$$

and

$$\int_{\Omega} |U|^p \leq \frac{C_{\Omega,p}}{1 + \beta^{\frac{p}{2}}} \int_{\partial\Omega} |u|^p.$$

*Proof.* Since  $\partial\Omega$  is compact and  $\Omega$  is of class  $C^1$ , there exists maps  $\psi_i : B(0,1) \times (-1,1) \rightarrow \mathbb{R}^d$  that are diffeomorphism on their image which is geodesically convex such that  $\psi_i(B(0,1) \cap [0,1]) = \psi_i(B(0,1) \times (0,1)) \cap \Omega$ ,  $\psi_i(B(0,1) \times \{0\}) = \psi_i(B(0,1)) \cap \partial\Omega$  and  $\partial\Omega \subset \bigcup_{i=1}^{\ell} \psi_i(B(0,1/2) \times \{0\})$ . Moreover, there exist smooth functions  $\eta_1, \dots, \eta_\ell$  in  $C^\infty(\bar{\Omega})$  such that  $\text{supp } \eta_i \subset \psi_i(B(0,1/2) \times [0,1/2])$  and  $\sum_{i=1}^{\ell} \eta_i = 1$  in  $\Omega$ .

By Proposition 3.3, for every  $i \in \{1, \dots, \ell\}$ , there exists a function  $U_i \in W_{\psi_i^* A}^{1,p}(B(0, 1/2) \times [0, 1/2]) \cap C^\infty(B(0, 1/2) \times [0, 1/2])$  such that  $\text{Tr } U_i = u \circ \psi_i$  on  $B(0, 1/2) \times \{0\}$  and moreover,

$$\int_{B(0,1/2) \times [0,1/2]} |\nabla_{\psi_i^* A} U_i|^p \leq C \iint_{B(0,1) \times B(0,1)} \frac{|e^{i\mathcal{I}_{\psi_i^* A}(x,y)} u(\psi_i(y, 0)) - u(\psi_i(x, 0))|^p}{|y-x|^{d+p-1}} dx dy + C\beta^{\frac{p-1}{2}} \int_{B(0,1)} |u(\psi_i(x, 0))|^p dx$$

and

$$\int_{B(0,1/2) \times [0,1/2]} |U_i|^p \leq C \int_{B(0,1)} |u \circ \psi_i|^p.$$

Using Lemma 6.1 and Lemma 6.2, we derive that

$$\int_{\psi_i(B(0,1/2) \times [0,1/2])} |\nabla_A(U_i \circ \psi_i^{-1})|^p \leq C \int_{\psi_i(B(0,1)) \times \psi_i(B(0,1))} \frac{|e^{i\mathcal{I}_A^{\partial\Omega}(x,y)} u(y) - u(x)|^p}{|y-x|^{d+p-1}} dx dy + \beta^{\frac{p-1}{2}} \int_{\psi_i(B(0,1) \times \{0\})} |u|^p$$

and

$$\int_{\psi_i(B(0,1/2) \times [0,1/2])} |U_i|^p \leq \int_{\psi_i(B(0,1) \times \{0\})} |u|^p.$$

We define now  $U \triangleq \sum_{i=1}^{\ell} \eta_i U_i \circ \psi_i^{-1}$ . We have by the Leibnitz rule for covariant derivatives

$$\nabla_A U = \sum_{i=1}^{\ell} ((U_i \circ \psi_i^{-1}) D\eta_i + \eta_i \nabla_A(U_i \circ \psi_i^{-1})),$$

and we conclude.  $\square$

Proposition 6.4 and Proposition 6.5 imply the following semi-classical estimates.

**Proposition 6.6.** *For every  $A \in C^1(\bar{\Omega}, \wedge^1 \mathbb{R}^d)$ , there exists a positive constant  $C$  such that for every  $\varepsilon > 0$ , we have*

$$\begin{aligned} & C^{-1} \left( \iint_{\substack{(x,y) \in \Omega \times \Omega \\ d_{\partial\Omega}(y,x) \leq \text{inj}_{\partial\Omega}}} \frac{\varepsilon^p |e^{i\mathcal{I}_A^{\partial\Omega}(x,y)/\varepsilon} u(y) - u(x)|^p}{|y-x|^{d+p-1}} dx dy + (\varepsilon^p + \varepsilon^{\frac{p+1}{2}}) \int_{\partial\Omega} |u|^p \right) \\ & \leq \inf \left\{ \int_{\Omega} |\varepsilon D U + i A U|^p + (\varepsilon^p + \varepsilon^{\frac{p}{2}}) |U|^p; U \in W_A^{1,p}(\Omega) \text{ and } \text{Tr}_{\partial\Omega} U = u \right\} \\ & \leq C \left( \iint_{\substack{(x,y) \in \Omega \times \Omega \\ d_{\partial\Omega}(y,x) \leq \text{inj}_{\partial\Omega}}} \frac{\varepsilon^p |e^{i\mathcal{I}_A^{\partial\Omega}(x,y)/\varepsilon} u(y) - u(x)|^p}{|y-x|^{d+p-1}} dx dy + (\varepsilon^p + \varepsilon^{\frac{p+1}{2}}) \int_{\partial\Omega} |u|^p \right). \end{aligned}$$

## 7. INTERPOLATION OF MAGNETIC SPACES

We define for every  $p \in [1, +\infty)$  and  $\gamma \in (0, +\infty)$ , the functional space [32, Definition 1.8.1/1]

$$\mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d) = \left\{ U : (0, +\infty) \rightarrow (W_A^{1,p}(\mathbb{R}^d) + L^p(\mathbb{R}^d)); \right.$$

$$\left. U \text{ is weakly differentiable and } \|U\|_{\mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d)} < +\infty \right\},$$

where

$$\|U\|_{\mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d)} \triangleq \left( \int_0^{+\infty} (\|U(t)\|_{W_A^{1,p}(\mathbb{R}^d)}^p + \|U'(t)\|_{L^p(\mathbb{R}^d)}^p) t^\gamma dt \right).$$

For every  $T \in (0, +\infty)$  and  $U \in \mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d)$ , one has  $U \in C([0, T], W_A^{1,p}(\mathbb{R}^d) + L^p(\mathbb{R}^d))$  [32, Lemma 1.8.1]. In particular, the corresponding trace space can be defined by [32, Definition 1.8.1/2]

$$(7.1) \quad \mathfrak{T}_{A,\gamma}^{1,p} \triangleq \{U(0); U \in \mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d)\}.$$

By a classical result in interpolation theory (see for example [32, Theorem 1.8.2]), we have if  $p \in [1, +\infty)$  and  $s \in (0, 1)$ ,

$$(7.2) \quad \mathfrak{T}_{A,(1-s)p-1}^{1,p}(\mathbb{R}^d) = (W_A^{1,p}(\mathbb{R}^d), L^p(\mathbb{R}^d))_{s,p},$$

where the right-hand side denotes the real interpolation of order  $s$  and exponent  $p$  between the spaces  $W_A^{1,p}(\mathbb{R}^d)$  and  $L^p(\mathbb{R}^d)$  [32, Definition 1.3.2].

In order to characterize the trace space, we rely on the following equivalence, whose non-magnetic counterpart is classical [32, Lemma 2.9.1/2]

**Lemma 7.1.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . If  $A \in C(\mathbb{R}^d, \mathbb{R}^d)$  and if  $dA \in L^\infty(\mathbb{R}^d, \wedge^2 \mathbb{R}^d)$ , then*

$$\mathfrak{W}_{A,\gamma}^{1,p}(\mathbb{R}^d) = W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1}).$$

Here  $\bar{A} : \mathbb{R}_+^{d+1} \rightarrow \mathbb{R}^{d+1}$  is the natural extension of  $A$ , defined by  $\bar{A}(x, t) = (A(x), 0)$ . The equality of Lemma 7.1 is understood under the identification  $U(t)(x) = U(x, t)$ .

*Proof of Lemma 7.1.* We assume first that  $U \in W_{A,s}^{1,p}(\mathbb{R}_+^{d+1})$ . By Fubini's theorem, we have for almost every  $t$ ,  $U(t) \in W^{1,p}(\mathbb{R}^d)$ . If now  $\theta \in C_c^\infty((0, +\infty))$ , we have for every  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,

$$(7.3) \quad \int_{\mathbb{R}^d} \int_0^{+\infty} \theta' \varphi U = - \int_{\mathbb{R}^d} \int_0^{+\infty} \theta \varphi U'$$

and thus, in  $W_A^{1,p}(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ ,

$$\int_0^{+\infty} \theta' \varphi U = - \int_0^{+\infty} \theta \varphi U'.$$

We finally have

$$\int_0^{+\infty} (\|U(t)\|_{W_A^{1,p}(\mathbb{R}^d)}^p + \|U'(t)\|_{L^p(\mathbb{R}^d)}^p) t^\gamma dt \leq C_1 \iint_{\mathbb{R}_+^{d+1}} (|\nabla_{\bar{A}} U(x, t)|^p + |U(x, t)|^p) t^\gamma dt dx.$$

Conversely, if  $U \in \mathfrak{W}_{A,(1-s)p-1}^{1,p}$ , then (7.3) holds and similarly,

$$\int_0^{+\infty} \theta (\operatorname{div} \varphi - iA \cdot \varphi) U(t) = - \int_0^{+\infty} \theta \varphi \nabla_A U(t)$$

Hence, by the density of tensor products, we obtain that  $U \in W_{\operatorname{loc}}^{1,1}(\mathbb{R}_+^{d+1})$  and  $\nabla_A U(t, x) = (\nabla_A(U(t))(x), U'(t))$ .  $\square$

We obtain from the previous results the following characterization of the spaces by interpolation.

**Theorem 7.2.** *Let  $d \geq 1$ ,  $0 < s < 1$  and  $1 \leq p < +\infty$ . If  $A \in C(\mathbb{R}^d, \mathbb{R}^d)$  and if  $dA \in L^\infty(\mathbb{R}^d, \wedge^2 \mathbb{R}^d)$ , then*

$$W_A^{s,p} = (W_A^{1,p}(\mathbb{R}^d), L^p(\mathbb{R}^d))_{s,p},$$

*Proof.* By Theorem 4.3,  $W_A^{s,p}(\mathbb{R}^d)$  is the trace space of  $W_{A,(1-s)p-1}^{1,p}(\mathbb{R}^d)$ . By Lemma 7.1, this latter space coincides with  $\mathfrak{W}_{A,(1-s)p-1}^{1,p}(\mathbb{R}^d)$  whose trace space  $\mathfrak{T}_A^{s,p}(\mathbb{R}^d)$  defined in (7.1) is the required interpolation space by (7.2).  $\square$

## 8. EXTENSION FROM A HALF-SPACE

Finally, we obtain a result about the extension from half-space to the whole space of functions in magnetic Sobolev spaces. Set, for  $\gamma \in \mathbb{R}$ ,

$$W_{A,\gamma}^{1,p}(\mathbb{R}^{d+1}) \triangleq \left\{ U \in W_{\text{loc}}^{1,1}(\mathbb{R}^{d+1}); \|U\|_{W_{A,\gamma}^{1,p}(\mathbb{R}^{d+1})} < +\infty \right\},$$

where

$$\|U\|_{W_{A,\gamma}^{1,p}(\mathbb{R}^{d+1})} \triangleq \left( \iint_{\mathbb{R}_+^{d+1}} (|\nabla_A U(x,t)|^p + |U(x,t)|^p) |t|^\gamma dx dt \right)^{\frac{1}{p}}.$$

**Theorem 8.1.** *Let  $d \geq 1$ ,  $-1 < \gamma < p - 1$  and  $1 \leq p < +\infty$ . There exists a constant  $C > 0$  such that for every  $A \in C^1(\mathbb{R}^{d+1}, \wedge^1 \mathbb{R}^{d+1})$  such that  $dA$  is bounded and every  $U \in W_{A,\gamma}^{1,p}(\mathbb{R}_+^{d+1})$ , there exists  $\bar{U} \in W_{A,\gamma}^{1,p}(\mathbb{R}^{d+1})$  such that  $\bar{U} = U$  on  $\mathbb{R}_+^{d+1}$ . Moreover, if  $\beta \geq \|dA\|_{L^\infty(\mathbb{R}^{d+1})}$ ,*

$$\iint_{\mathbb{R}^{d+1}} (|\nabla_A \bar{U}(x,t)|^p + \beta^{\frac{p}{2}} |\bar{U}(x,t)|^p) t^\gamma dx dt \leq C \iint_{\mathbb{R}_+^{d+1}} (|\nabla_A U(x,t)|^p + \beta^{\frac{p}{2}} |U(x,t)|^p) t^\gamma dx dt.$$

*Proof of Theorem 8.1.* This follows from Theorem 4.3 on  $\mathbb{R}_+^{d+1}$  and Proposition 3.1 on  $\mathbb{R}_-^{d+1}$  with  $s = 1 - \frac{\gamma+1}{p}$ .  $\square$

*Remark 8.2.* A natural strategy to prove Theorem 8.1 would be to define the extension  $\bar{U}$  by reflection: for every  $(x,t) \in \mathbb{R}^d \times (-\infty, 0)$ , we would define  $\bar{U}(x,t) = U(x,-t)$ . The computation of the covariant derivative would give

$$\nabla_A \bar{U}(x,t) = R \nabla_A U(x,-t) + i(A(x,t) - RA(x-t))U(x,-t),$$

where  $R$  is the orthogonal reflection with respect to the hyperplane  $\mathbb{R}^d \times \{0\}$ . The approach would thus only work when  $A$  is invariant under the pull-back by  $R$ ; this would imply the same invariance of the magnetic field  $dA$  and implies that even up to gauge transformations, it is not possible to cover a significant class of magnetic fields.

## APPENDIX A. ALTERNATIVE MAGNETIC GAGLIARDO SEMINORMS

A fractional Gagliardo seminorm defined by an integral involving the mid-point has been proposed in the litterature (see [1; 2; 5, §2; 8, §2; 12; 15; 18, §2; 25; 27; 31; 35, §2]):

$$(A.1) \quad \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{iA(\frac{x+y}{2})[y-x]} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}};$$

another natural candidate could be the integral involving boundary points

$$(A.2) \quad \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{\frac{i}{2}(A(x)[y-x] + A(y)[y-x])} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}}.$$

Whereas (1.4) enjoyed a gauge-invariance property, this is not the case for (A.1) or (A.2).

The formulas (1.4), (A.1) and (A.2) are in fact particular cases of the following general semi-norm

$$(A.3) \quad \|u\|_{\dot{W}_{A,\mu}^{s,p}(\mathbb{R}^d)} = \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A^\mu}^{x,y}} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}},$$

where  $\mu$  is a given finite measure on the interval  $[0, 1]$  and where the potential  $\mathcal{I}_{A^\mu}^\mu$  of  $A$  with respect to the measure  $\mu$  is defined by the following variant of (2.4)

$$\mathcal{I}_{A^\mu}^\mu(x,y) = \int_0^1 A((1-t)x + ty) \cdot (y-x) d\mu(t).$$

The seminorm defined in (1.4), (A.1) and (A.2) correspond respectively to a restriction of the Lebesgue measure  $\mu = \mathcal{L}^1 \llcorner [0, 1]$ , a Dirac measure at the centre  $\mu = \delta_{1/2}$  and the average of Dirac measures at the endpoint  $\mu = \frac{\delta_0 + \delta_1}{2}$ .

**Proposition A.1.** *Let  $k \in \mathbb{N}$  and let  $\mu_1, \mu_2$  be measures on  $[0, 1]$ . If for every  $j \in \{0, \dots, k-1\}$ ,*

$$\int_0^1 t^j d\mu_1(t) = \int_0^1 t^j d\mu_2(t),$$

then

$$\left| \|u\|_{W_{A, \mu_2}^{s,p}(\mathbb{R}^d)} - \|u\|_{W_{A, \mu_1}^{s,p}(\mathbb{R}^d)} \right| \leq C \|D^k A\|_{L^\infty(\mathbb{R}^d)}^{\frac{s}{k+1}} \|u\|_{L^p(\mathbb{R}^d)}.$$

The measures corresponding to the semi-norms of (1.4), (A.1) and (A.2) all satisfy the assumptions of Proposition A.1 with  $k = 0$ ,  $k = 1$  and  $k = 2$ , and the heterogeneous spaces coincide thus as soon as either  $A$  is bounded or its second or first derivative is bounded. This means in particular that if  $k = 1$ , the estimates in the present work involving  $\|dA\|_{L^\infty}$  and our semi-norm given by (1.4), have counterparts involving  $\|DA\|_{L^\infty}$  and either (A.1) and (A.2); the latter quantities are not gauge invariant.

In the particular case where  $A$  is an affine function, then Proposition A.1 holds with  $k = 2$  and  $D^2 A = 0$ ; the norms defined by (1.4), (A.1) and (A.2) are then identical.

Higher moment identities in the assumption of Proposition A.1 induce lower powers in the dependence on derivatives of  $A$ , which can be relevant in semi-classical analyses. The exponent in the moment condition can be increased by using a measure  $\mu$  such that more moment coincide. For instance setting  $\mu = \frac{1}{6}\delta_0 + \frac{2}{3}\delta_{1/2} + \frac{1}{6}\delta_1$ , corresponding to Simpson's quadrature rule, would give estimates of (1.4) up to a term of the order  $\|D^k A\|_{L^p}^{\frac{sp}{k+1}}$  for  $k \in \{0, \dots, 3\}$ .

*Proof of Proposition A.1.* We have for every  $x, y \in \mathbb{R}^d$ , by the triangle inequality

$$\begin{aligned} & \left| |e^{i\mathcal{I}_{A''}^{\mu_2}(x,y)} u(y) - u(x)| - |e^{i\mathcal{I}_{A''}^{\mu_1}(x,y)} u(y) - u(x)| \right| \\ & \leq |e^{i(\mathcal{I}_{A''}^{\mu_2}(x,y) - \mathcal{I}_{A''}^{\mu_1}(x,y))} u(y) - u(y)| = |e^{i\mathcal{I}_{A''}^{\mu_2 - \mu_1}(x,y)} u(y) - u(y)|. \end{aligned}$$

By integration and Minkowski's inequality, this yields

$$\begin{aligned} & \left| \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}^{\mu_2}(x,y)} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}} \right. \\ & \quad \left. - \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}^{\mu_1}(x,y)} u(y) - u(x)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}} \right| \\ & \leq \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}^{\mu_2 - \mu_1}(x,y)} - 1|^p |u(y)|^p}{|y-x|^{d+sp}} dx dy \right)^{\frac{1}{p}}. \end{aligned}$$

By our assumption on the moments of the measures  $\mu_1$  and  $\mu_2$ , there exists a constant  $C_1$  depending only on the measure  $\mu_2 - \mu_1$  such that

$$|\mathcal{I}_{A''}^{\mu_2 - \mu_1}(x, y)| \leq C_1 \|D^k A\| |y-x|^{k+1},$$

and thus for every  $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \frac{|e^{i\mathcal{I}_{A''}^{\mu_2 - \mu_1}(x,y)} - 1|^p}{|y-x|^{d+sp}} dx \leq C_2 \|D^k A\|_{L^\infty}^{\frac{sp}{k+1}};$$

the conclusion then follows.  $\square$

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