Discrete spectrum of perturbed Dirac systems with real and periodic coefficients

Boris Buffoni

Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland (MS received 16 November 1989. Revised MS received 12 February 1990)

Synopsis

This paper deals with the number of eigenvalues which appear in the gaps of the spectrum of a Dirac system with real and periodic coefficients when the coefficients are perturbed. The main results provide an upper bound and a condition under which exactly one eigenvalue appears in a given gap.

Introduction

Let us consider the differential expression

\[ \sigma y(x) = -y''(x) + q(x)y(x), \]

where \( y \) is a complex valued function defined on \( \mathbb{R} \) and \( q: \mathbb{R} \to \mathbb{R} \) is periodic with period \( a > 0 \) and locally absolutely integrable.

The maximal operator \( S \) generated by \( \sigma \) on \( \mathbb{R} \) [7] is self-adjoint and is called the Hill's operator. Its spectrum \( \sigma(S) \subset \mathbb{R} \) is purely continuous, bounded from below but unbounded from above, and it is a locally finite union of closed intervals of positive length. In the following, we shall suppose that \( \sigma(S) \) has an infinity of gaps; this is so, for example, if \( q \in L^2_{\text{loc}}(\mathbb{R}) \) and is not analytic [5].

Rofe-Beketov [3, 4] studied the perturbed Hill's operator \( S \), which is the maximal operator generated on \( \mathbb{R} \) by the differential expression

\[ \tilde{\sigma} y(x) = -y''(x) + \{q(x) + \Delta q(x)\}y(x), \]

where \( \Delta q: \mathbb{R} \to \mathbb{R} \) is such that \( |\Delta q(x)| (1 + |x|) \) is integrable on \( \mathbb{R} \). This is a self-adjoint operator with the same essential spectrum \( \sigma_e(S) \) as \( S \). He proved that there is only a finite number of eigenvalues of \( \tilde{S} \) in each gap and at most two eigenvalues in each sufficiently remote gap; moreover, there is exactly one eigenvalue in each sufficiently remote gap if the following additional condition is satisfied: \( \int \Delta q(x) \, dx \neq 0 \).

The purpose of this paper is to prove analogous results for Dirac systems. Let \( \tau \) be the differential expression

\[ \tau u(x) = R(x)^{-1}\left( \begin{array}{c} 0 \\ 1 \end{array} \right)u'(x) + P(x)u(x), \]

where \( u \) is a \( \mathbb{C}^2 \)-valued function defined on \( \mathbb{R} \); \( P \) and \( R \) are symmetric \( 2 \times 2 \) matrices, with locally absolutely integrable real entries which are periodic with period \( a > 0 \); \( R \) is positive definite almost everywhere; and let \( L^2([c, d], R) \) be
the Hilbert space defined by

\[ L^2([c, d], R) = \left\{ u: [c, d] \rightarrow \mathbb{C}^2 \mid \int_c^d (R(x)u(x), u(x)) \, dx < \infty \right\} \]

and

\[ (u, v) = \int_c^d (R(x)u(x), v(x)) \, dx, \]

where \((., ., ., .)\) denotes the usual scalar product in \(\mathbb{C}^2\) and \(-\infty \leq c < d \leq +\infty\) (we do not distinguish between two functions which are equal almost everywhere). The maximal operator \(T\) generated by \(\tau\) and defined in \(L^2(\mathbb{R}, R)\) is self-adjoint, its spectrum \(\sigma(T)\) is purely continuous, unbounded from above and below, and it is a locally finite union of closed intervals of positive length [7, theorem 12.5]. In the following, we shall suppose that \(\sigma(T)\) has at least one gap; for example, if

\[ a = 2, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P|_{[0,1]} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P|_{[1,2]} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \]

there is an infinity of gaps [7, chap. 17.G].

We shall examine the perturbed operator \(\tilde{T}\), which is the maximal operator generated on \(\mathbb{R}\) by the differential expression

\[ \tilde{\tau}u(x) = R(x)^{-1}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}u'(x) + \{P(x) + \Delta P(x)\}u(x) \right\}, \]

where \(\Delta P(x)\) is a symmetric \(2 \times 2\) matrix, with absolutely integrable real entries, and whose support satisfies \(\text{supp} (\Delta P) \subset [A, B] (-\infty < A \leq B < +\infty)\). The operator \(\tilde{T}\) is self-adjoint and \(\sigma_e(\tilde{T}) = \sigma_e(T)\); this can be proved by the method of decomposition [1].

Let \([\mu, \nu]\) be a gap of \(\sigma_e(\tilde{T})\), let \(r_2(x), p_2(x), \delta p_2(x)\) be the largest eigenvalues of respectively \(R(x), P(x), \Delta P(x)\), and let \(r_1(x), p_1(x), \delta p_1(x)\) be the corresponding lowest eigenvalues. We shall show that if for \(N \in \mathbb{N}\),

\[ \int_A^B \{(|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x))\} \, dx \leq N\pi, \]

then there are at most \(N + 1\) eigenvalues of \(\tilde{T}\) in \([\mu, \nu]\), and if

\[ \int_A^B \{\max (|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|\} \, dx \leq \pi/2, \]

\[ \delta p_1(x) = \delta p_2(x) \quad \text{almost everywhere on } \mathbb{R}, \]

\[ \delta p_2(x) \text{ is not equal almost everywhere to the null function}, \]

\[ \delta p_2(x) \geq 0 \quad \text{almost everywhere or} \quad \delta p_2(x) \leq 0 \quad \text{almost everywhere on } \mathbb{R}, \]

then there is exactly one eigenvalue in \([\mu, \nu]\).

But first we shall present an important tool, which is an adaptation of the oscillation theory for Dirac systems developed by Weidmann [7, chap. 16].
1. Oscillation theory for Dirac systems defined on $\mathbb{R}$ and which are in the limit point case at $-\infty$ and $+\infty$

Let $\tau$ be the differential expression

$$\tau u(x) = R(x)^{-1} \left\{ 2 \begin{pmatrix} 0 & q(x) \\ -q(x) & 0 \end{pmatrix} u'(x) + \begin{pmatrix} 0 & q'(x) \\ -q'(x) & 0 \end{pmatrix} u(x) + P(x)u(x) \right\},$$

where $u$ is a $C^2$-valued function defined on $\mathbb{R}$; $P$ and $R$ are symmetric $2 \times 2$ matrices, with locally absolutely integrable real entries; $R$ is positive definite almost everywhere; $q$ is a locally absolutely continuous real valued function and for all values of $x \in \mathbb{R}$: $q(x) > 0$. If $u$ is a non-trivial real solution of $\tau u = \lambda u$, we introduce the transformation

$$u(x) = \rho(x) \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}, \quad \rho(x) > 0,$$

where $\rho$ and $\theta$ are continuous and $\theta$ is defined up to $2k\pi$. If $\theta$ is completely defined (for instance, if we know its value at a given $x_0$), we shall call it a determination of the angular part of $u$.

It is easy to check that

$$\theta'(x) = \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}, \quad \theta'(x) = \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix},$$

with

$$G(x) = \frac{1}{2q(x)} (\lambda R(x) - P(x)).$$

We write $\theta(\lambda, \alpha, c, x)(\lambda, \alpha, c \text{ in } \mathbb{R})$ for the angular part which

(i) corresponds to the solution $v(x) = (v_1(x), v_2(x))$ of $\tau u = \lambda u$, satisfying $v_1(c) = \cos (\alpha)$ and $v_2(c) = \sin (\alpha)$;

(ii) is such that $\theta(\lambda, \alpha, c, c) = \alpha$.

**Proposition 1.1.** If the maximal operator $T$ generated by $\tau$ on $\mathbb{R}$ is self-adjoint and $-\infty < \mu < \lambda < +\infty$, then

$$\dim R(E_T(\lambda-) - E_T(\mu)) \leq \lim \inf_{n \to \infty} \left\lfloor \frac{\theta(\lambda, \alpha_n, c_n, d_n) - \theta(\mu, \alpha_n, c_n, d_n)}{\pi} \right\rfloor,$$

where $(\alpha_n) \subset \mathbb{R}$, $c_n \to -\infty$, $d_n \to +\infty$, $E_T(.)$ is the right continuous spectral resolution of $T$ and $\lfloor x \rfloor = \max \{k \in \mathbb{Z} \mid k \leq x\}$.

**Proof.** Let $\delta > 0$ be such that $\mu + \delta$ and $\lambda - \delta$ are not eigenvalues of $T$ and $\mu + \delta < \lambda - \delta$, let $\beta_n$ be defined by $\beta_n = \theta(\mu + \delta, \alpha_n, c_n, d_n)$ and let us define the self-adjoint operators $B_n$, $O_n$ and $T_n$ as follows:

$$B_n: \begin{cases} D(B_n) = L^2([c_n, d_n[, R) \to L^2([c_n, d_n[, R), \\ B_nu = \tau u, \end{cases}$$

with

$$D(B_n) = \{ u \in L^2([c_n, d_n[, R) \mid u \text{ is loc. abs. cont., } \tau u \in L^2([c_n, d_n[, R) \text{ and}$$

$$\sin (\alpha_n)u_1(c_n) - \cos (\alpha_n)u_2(c_n) = \sin (\beta_n)u_1(d_n) - \cos (\beta_n)u_2(d_n) = 0 \},$$

$O_n$ is the null operator on $L^2(-\infty, c_n[, R) \oplus L^2([d_n, +\infty[, R)$ and $T_n = B_n \oplus O_n$. 


The sequence \((T_n)\) converges to \(T\) in the sense of the strong resolvent convergence and therefore
\[
\dim R(E_T(\lambda - \delta) - E_T(\mu + \delta)) \leq \liminf_{n \to \infty} \dim R(E_T(\lambda - \delta) - E_{T_n}(\mu + \delta))
\]
\[
= \liminf_{n \to \infty} \dim R(E_{B_n}(\lambda - \delta) - E_{B_n}(\mu + \delta))
\]
(see [6, theorems 9.16.(i) and 9.19]). We have
\[
\dim R(E_{B_n}(\lambda - \delta) - E_{B_n}(\mu + \delta))
\]
\[
= \text{card} \left( \{ t \in [\mu + \delta, \lambda - \delta] \mid \theta(t, \alpha_n, c_n, d_n) = \beta_n \mod \pi \} \right),
\]
\[
\leq \left[ \frac{\theta(\lambda, \alpha_n, c_n, d_n) - \theta(\mu, \alpha_n, c_n, d_n)}{\pi} \right]
\]
because \(\theta(t, \alpha_n, c_n, d_n)\) is increasing in \(t\) [7, theorem 16.1]. Therefore,
\[
\dim R(E_T(\lambda - ) - E_T(\mu)) \leq \liminf_{\delta \to 0} \dim R(E_T(\lambda - \delta) - E_T(\mu + \delta))
\]
\[
\leq \liminf_{n \to \infty} \left[ \frac{\theta(\lambda, \alpha_n, c_n, d_n) - \theta(\mu, \alpha_n, c_n, d_n)}{\pi} \right].
\]

**Proposition 1.2.** If \(T\) is any self-adjoint operator generated by \(\tau\) on \(\mathbb{R}\) and \(-\infty < \mu < \lambda < +\infty\), then
\[
\dim R(E_T(\lambda) - E_T(\mu)) \leq \sup_{c < d, \alpha \in \mathbb{R}} \left[ \frac{\theta(\lambda, \alpha, c, d) - \theta(\mu, \alpha, c, d)}{\pi} \right] - 1.
\]

**Proof.** Let us choose \(c, d, \alpha\) in \(\mathbb{R}\), and let us suppose that \(c < d\) and
\[
n := \left[ \frac{\theta(\lambda, \alpha, c, d) - \theta(\mu, \alpha, c, d)}{\pi} \right] \geq 2.
\]
We introduce the operator
\[
B : \{ D(B) \subset L^2([c, d], R) \mid u \text{ is loc. abs. cont., } \tau u \in L^2([c, d], R) \}
\]
\[
Bu = \tau u,
\]
with
\[
/D(B) = \{ u \in L^2([c, d], [R]) \mid u \text{ is loc. abs. cont., } \tau u \in L^2([c, d], [R]) \}
\]
\[
\sin (\alpha)u_1(c) - \cos (\alpha)u_2(c) = \sin (\beta)u_1(d) - \cos (\beta)u_2(d) = 0,
\]
where \(\beta = \theta(\mu, \alpha, c, d)\). Since \(\dim R(E_B(\lambda) - E_B(\mu)) = \text{card} \left( \{ t \in [\mu, \lambda] \mid \theta(t, \alpha, c, d) = \beta \mod \pi \} \right) = n + 1\), there exists a subspace \(M \subset R(E_B(\lambda) - E_B(\mu))\) such that (i) \(\dim (M) \geq n - 1\), (ii) for all values of \(u\) in \(M\), \(u(c) = u(d) = 0\) and \(\|B - (\lambda + \mu)/2\| u\| \leq \| (\lambda - \mu)/2 \| u\|\). We can consider \(M\) as a subspace of \(D(T)\) and so we have \(\dim R(E_T(\lambda) - E_T(\mu)) \geq n - 1\) (if we had \(\dim R(E_T(\lambda) - E_T(\mu)) < n - 1\), then there would exist \(f \in M\) such that \(f \neq 0\), \(f \perp R(E_T(\lambda) - E_T(\mu))\) and \(\|T - (\lambda + \mu)/2\| f\| > \| (\lambda - \mu)/2 \| f\|\)).

\[
\square
\]
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**Proposition 1.3.** Let \( u \) and \( \bar{u} \) be two non-trivial real solutions of \( ru = \lambda u \) (\( \lambda \) in \( \mathbb{R} \)) and let \( \theta \) and \( \bar{\theta} \) be any two determinations of the corresponding angular parts. If there exists \( x_0 \) in \( \mathbb{R} \) and \( k \in \mathbb{Z} \) such that \( \bar{\theta}(x_0) - \theta(x_0) \in [k \pi, (k + 1)\pi] \), then for all values of \( x \in \mathbb{R} \): \( \bar{\theta}(x) - \theta(x) \in [k \pi, (k + 1)\pi] \).

**Proof.** We have

\[
\sigma'(x) = \bar{\theta}'(x) - \theta'(x) = g_{11}(x) \cos^2 \bar{\theta}(x) + g_{22}(x) \sin^2 \bar{\theta}(x) \\
+ 2g_{12}(x) \sin \bar{\theta}(x) \cos \bar{\theta}(x) \\
- g_{11}(x) \cos^2 \theta(x) - g_{22}(x) \sin^2 \theta(x) - 2g_{12}(x) \sin \theta(x) \cos \theta(x) \\
= \{g_{22}(x) - g_{11}(x)\} \{\sin^2 \bar{\theta}(x) - \sin^2 \theta(x)\} + g_{12}(x) \{\sin 2\bar{\theta}(x) - \sin 2\theta(x)\} \\
= \{g_{22}(x) - g_{11}(x)\} \sin \{\bar{\theta}(x) + \theta(x)\} \sin \sigma(x) \\
+ 2g_{12}(x) \cos \{\bar{\theta}(x) + \theta(x)\} \sin \sigma(x),
\]

where \( \sigma(x) = \bar{\theta}(x) - \theta(x) \). This differential equation verifies the local existence and uniqueness theorem. Since \( \sigma = k\pi \) and \( \sigma = (k + 1)\pi \) are solutions and \( \sigma(x_0) \in [k \pi, (k + 1)\pi] \), we have \( \sigma(x) \in [k \pi, (k + 1)\pi] \) for all \( x \) in \( \mathbb{R} \). \( \square \)

**2. Dirac systems with periodic and real coefficients**

We suppose that \( q = \frac{1}{2} \) and that \( P \) and \( R \) have the period \( a > 0 \); \( \tau \) becomes

\[
\tau u(x) = R(x)^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u'(x) + P(x)u(x).
\]

For \( \lambda \) in \( \mathbb{R} \), let us introduce the fundamental system of solutions of \( \tau u(x) = \lambda u(x) \):

\( \varphi(x, \lambda), \psi(x, \lambda) \), satisfying

\[
\varphi(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

We have

\[
W[\varphi, \psi](x, \lambda) := \begin{vmatrix} \varphi_1(x, \lambda) & \psi_1(x, \lambda) \\ \varphi_2(x, \lambda) & \psi_2(x, \lambda) \end{vmatrix} = 1.
\]

We introduce the discriminant, which is the real valued function defined by

\( D(\lambda) = \varphi_1(a, \lambda) + \psi_2(a, \lambda) \) \((\lambda \in \mathbb{R}) \).

For \( \lambda \in \mathbb{R} \) such that \( |D(\lambda)| \geq 2 \), let \( \rho_1(\lambda) \) and \( \rho_2(\lambda) \) in \( \mathbb{R} \) be the two roots of \( \rho^2 - D(\lambda)\rho + 1 \) with \( |\rho_1(\lambda)| \leq 1 \leq |\rho_2(\lambda)| \), and let \( k(\lambda) \) in \( \mathbb{C} \) satisfy \( \exp \{ak(\lambda)\} = \rho_2(\lambda) \). For \( i = 1, 2 \), there exists a real solution \( e_i(x, \lambda) \) of \( \tau u(x) = \lambda u(x) \) such that

\[
e_i(x + a, \lambda) = \rho_i(\lambda)e_i(x, \lambda),
\]

and if we define \( z_1 \) and \( z_2 \) by

\[
e_1(x, \lambda) = \exp \{-k(\lambda)x\}z_1(x, \lambda) \quad \text{and} \quad e_2(x, \lambda) = \exp \{k(\lambda)x\}z_2(x, \lambda),
\]

then \( z_i(x + a, \lambda) = z_i(x, \lambda) \) for all \( x \) in \( \mathbb{R} \) and \( i = 1, 2 \).

If \( \rho_1(\lambda) \neq \rho_2(\lambda) \), then \( e_1(x, \lambda) \) and \( e_2(x, \lambda) \) can be chosen linearly independent; the same is possible if \( \rho_1(\lambda) = \rho_2(\lambda) = \pm 1 \) and \( \varphi_2(a, \lambda) = \psi_1(a, \lambda) = 0 \). If
\[ \rho_1(\lambda) = \rho_2(\lambda)(= \pm 1) \] and \(|\varphi_2(a, \lambda)| + |\psi_1(a, \lambda)| > 0\), then we can choose \(e_i(x, \lambda)\) such that \(e_1(x, \lambda) = e_2(x, \lambda) \neq (0, 0)\) (for all values of \(x \in \mathbb{R}\)), and there exists a solution \(y(x)\) of \(\tau u = \lambda u\), linearly independent of \(e_i(x, \lambda)\), such that \(\lim_{x \to \infty} |y(x)| = \lim_{x \to \infty} |y(x)| = +\infty\).

We can take for example
\[
e_i(x, \lambda) = \psi_1(a, \lambda)\varphi(x, \lambda) + (\rho_1(\lambda) - \varphi_1(a, \lambda))\psi(x, \lambda), \quad \text{or} \\
e_i(x, \lambda) = (\rho_1(\lambda) - \varphi_2(a, \lambda))\varphi(x, \lambda) + \psi_2(a, \lambda)\psi(x, \lambda),
\]
but these functions can be null \((i = 1, 2)\). Note that they are analytic in \(\lambda\) on \(\{\lambda \in \mathbb{R} \mid |D(\lambda)| > 2\}\) for every \(x \in \mathbb{R}\). The reader is referred to [2] and [7] for more information.

The maximal operator defined on \(\mathbb{R}\) by \(\tau\) is self-adjoint and its spectrum is equal to \(\{\lambda \in \mathbb{R} \mid |D(\lambda)| \leq 2\}\). Let \([\mu, \nu]\) be a gap of its spectrum and let \(e_\mu\) and \(e_\nu\) be two non-trivial, real and periodic solutions of, respectively, \(\tau u = \mu u\) and \(\tau u = \nu u\). We shall denote any two determinations of the corresponding angular parts by \(\theta_\mu\) and \(\theta_\nu\). Let \(c < d\) be in \(\mathbb{R}\) such that \(d - c\) is in a \(\mathbb{Z}\).

**Proposition 2.1.** (1) \(\theta_\nu(d) - \theta_\mu(d) = \theta_\nu(c) - \theta_\mu(c)\);
(2) if \(\theta\) is the determination of the angular part of a non-trivial real solution of \(\tau u = vu\) such that \(\theta(c) = \theta_\mu(c)\), then \(\theta(d) - \theta_\mu(d) \in ]0, \pi[\).

**Proof.** (1) Since \(\theta_\mu(x)\) and \(\theta_\nu(x)\) are the angular parts of periodic or semi-periodic functions of period or semi-period \(a\), there exists \(k\) in \(\mathbb{Z}\) such that \(\{\theta_\nu(d) - \theta_\mu(d)\} = \{\theta_\nu(c) - \theta_\mu(c)\} = k\pi\).

Case (i). Let us suppose that \(k > 0\). For \(n \in \mathbb{N}\), set \(c_n = c - (d - c)n\) and \(d_n = d + (d - c)n\). Then \(\{\theta_\nu(d_n) - \theta_\mu(d_n)\} = \{\theta_\nu(c_n) - \theta_\mu(c_n)\} = (2n + 1)k\pi\). Let \(\theta_n\) be the determination of the angular part of a non-trivial real solution of \(\tau u = vu\) such that \(\theta_n(c_n) = \theta_\mu(c_n)\). We have
\[
\theta_n(d_n) - \theta_\mu(d_n) = (\{\theta_n(d_n) - \theta_\nu(d_n)\} - \{\theta_n(c_n) - \theta_\nu(c_n)\}) \\
+ (\{\theta_\nu(d_n) - \theta_\mu(d_n)\} - \{\theta_\nu(c_n) - \theta_\mu(c_n)\}).
\]
By Proposition 1.3, the first term belongs to \(] - \pi, \pi[\) and therefore
\[
\theta_n(d_n) - \theta_\mu(d_n) \in ](2n + 1)k\pi - \pi, (2n + 1)k\pi + \pi[.
\]
By Proposition 1.2, there is an infinity of eigenvalues of \(T\) in \([\mu, \nu]\). This assertion being false, we have proved that \(k \leq 0\).

Case (ii). Let us suppose that \(k < 0\). Then \(\theta_n(d_n) - \theta_\mu(d_n) < 0\) for \(n \in \mathbb{N}\). Hence we have a contradiction with [7, theorem 16.1].

(2) There exists \(k\) in \(\mathbb{Z}\) such that \(\theta(c) - \theta_\mu(c) = \theta_\nu(c) - \theta_\nu(c) = \theta(d) - \theta_\nu(d) \in [k\pi, (k + 1)\pi[\). By Proposition 1.3, \(\theta(d) - \theta_\nu(d) \in [k\pi, (k + 1)\pi[\). Thus \(\theta(d) - \theta_\mu(d) = (\theta(d) - \theta_\nu(d)) - (\theta_\mu(d) - \theta_\nu(d)) \in ] - \pi, \pi[\).

By [7, theorem 16.1], \(\theta(d) > \theta_\mu(d)\) and therefore \(\theta(d) - \theta_\mu(d) \in ]0, \pi[\).
3. The main results

Let $\tau$ and $\breve{\tau}$ be as in the Introduction and let $]\mu, \nu[$ be a gap of $\sigma_c(\breve{T})$ (we suppose that there is at least one gap).

**Proposition 3.1.** If for $N \in \mathbb{N}$,
\[ \int_A^B \{(|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x))\} \, dx \leq N\pi, \]
then there are at most $(N + 1)$ eigenvalues of $\breve{T}$ in $]\mu, \nu[.

**Proof.** Let $e_\mu$ be a non-trivial, real and periodic solution of $\tau u = \mu u$, and let $\theta_\mu$ be any determination of its angular part. For $n \in \mathbb{N}$, we introduce the following notation: $\theta$ is the determination of the angular part of a real non-trivial solution of $\tau u = \mu u$ such that $\theta(-na) = \theta_\mu(-na)$; $\tilde{\theta}$ is the determination of the angular part of a real non-trivial solution of $\breve{\tau} u = \mu u$ such that $\tilde{\theta}(-na) = \tilde{\theta}_\mu(-na)$.

We have
\[ \theta_\mu'(x) = \left( (\mu R(x) - P(x)) \begin{pmatrix} \cos \theta_\mu(x) \\ \sin \theta_\mu(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}_\mu(x) \\ \sin \tilde{\theta}_\mu(x) \end{pmatrix} \right), \]
\[ \tilde{\theta}_\mu'(x) = \left( (\mu R(x) - P(x) - \Delta P(x)) \begin{pmatrix} \cos \tilde{\theta}_\mu(x) \\ \sin \tilde{\theta}_\mu(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}_\mu(x) \\ \sin \tilde{\theta}_\mu(x) \end{pmatrix} \right), \]
\[ \theta'(x) = \left( (\nu R(x) - P(x)) \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}(x) \\ \sin \tilde{\theta}(x) \end{pmatrix} \right), \]
\[ \tilde{\theta}'(x) = \left( (\nu R(x) - P(x) - \Delta P(x)) \begin{pmatrix} \cos \tilde{\theta}(x) \\ \sin \tilde{\theta}(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}(x) \\ \sin \tilde{\theta}(x) \end{pmatrix} \right), \]
and thus
\[ |\{\tilde{\theta}'(x) - \theta'(x)\} - \{\tilde{\theta}_\mu'(x) - \theta_\mu'(x)\}| \leq (|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x)). \]

Let us suppose that $n$ is such that $[A, B) \subset ]-na, na[$ and let $k$ be in $\mathbb{Z}$ such that $\tilde{\theta}_\mu(B) - \theta_\mu(B) \in [k\pi, (k + 1)\pi[$. We have
\[ |\{\tilde{\theta}(B) - \theta(B)\} - \{\tilde{\theta}_\mu(B) - \theta_\mu(B)\}| \leq \int_A^B |\{\tilde{\theta}'(x) - \theta'(x)\} - \{\tilde{\theta}_\mu'(x) - \theta_\mu'(x)\}| \, dx \leq N\pi \]
and therefore $\tilde{\theta}(B) - \theta(B) \in [(k - N)\pi, (k + N + 1)\pi[$. By Proposition 1.3, $\tilde{\theta}_\mu(na) - \theta_\mu(na) \in [k\pi, (k + 1)\pi[$ and $\tilde{\theta}(na) - \theta(na) \in [(k - N)\pi, (k + N + 1)\pi[$. Using $\theta(na) - \tilde{\theta}_\mu(na) \in ]0, \pi[$ (Proposition 2.1 (2)), we get
\[ \tilde{\theta}(na) - \tilde{\theta}_\mu(na) = \{\tilde{\theta}(na) - \theta(na)\} - \{\tilde{\theta}_\mu(na) - \theta_\mu(na)\} \]
\[ + \{\theta(na) - \tilde{\theta}_\mu(na)\} < (k + N + 1 - k + 1)\pi = (N + 2)\pi. \]
Letting $n$ tend to $+\infty$, the result now follows from Proposition 1.1 and the fact that the eigenvalues are of multiplicity one (for $\lambda \in ]\mu, \nu[$, a solution in $L^2(\mathbb{R}, R)$ of $\tilde{t}u = \lambda u$ is a multiple of $e_1(., \lambda)$ on $[B, \infty[$ and a multiple of $e_2(., \lambda)$ on $]-\infty, A[)$).

**Proposition 3.2.** If

$$\int_A^{B} \left\{ \max (|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)| \right\} \, dx \leq \pi/2,$$

$$\delta p_1(x) = \delta p_2(x) \quad \text{almost everywhere on } \mathbb{R},$$

$\delta p_2$ is not equal almost everywhere to the null function,

$$\delta p_2(x) \geq 0 \quad \text{almost everywhere or} \quad \delta p_2(x) \leq 0 \quad \text{almost everywhere on } \mathbb{R},$$

then there is exactly one eigenvalue of $\tilde{T}$ in $]\mu, \nu[$.

**Proof.** We shall adapt a method of Rofe-Beketov [4]. Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be as in Section 2 and let $\tilde{\varphi}(x, \lambda)$ and $\tilde{\psi}(x, \lambda)$ be two solutions of $\tilde{t}u(x) = \lambda u(x)$ satisfying

$$\tilde{\varphi}(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\psi}(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us introduce the two regular matrices

$$L(x, \lambda) = (\varphi(x, \lambda) \, \psi(x, \lambda)) \quad \text{and} \quad \tilde{L}(x, \lambda) = (\tilde{\varphi}(x, \lambda) \, \tilde{\psi}(x, \lambda)),$$

let $v$ and $w$ be two real solutions of $\tau u = \lambda u$ ($\lambda \in \mathbb{R}$ is fixed) and let $\tilde{v}$ and $\tilde{w}$ be two real solutions of $\tilde{t}u = \lambda u$, such that $v$ and $\tilde{v}$ are equal on $[B, +\infty[$, and $w$ and $\tilde{w}$ are equal on $]-\infty, A[.$

Using the method of variation of constants, we get

$$\tilde{w}(x) = w(x) + L(x, \lambda) \int_A^x \left\{ L^{-1}(t, \lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta P(t) \tilde{w}(t) \right\} \, dt,$$

$$v(x) = \tilde{v}(x) - \tilde{L}(x, \lambda) \int_B^x \left\{ \tilde{L}^{-1}(t, \lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta P(t) v(t) \right\} \, dt,$$

and using

$$v'(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L(x, \lambda) = v'(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L(t, \lambda)$$

and

$$\tilde{w}'(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(x, \lambda) = \tilde{w}'(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(t, \lambda),$$

we obtain

$$W[\tilde{v}, \tilde{w}] = W[v, w] + v' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\tilde{w} - w) - \tilde{w}' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\tilde{v} - v)$$

$$= W[v, w] - \int_A^B (\Delta P(t) v(t), \tilde{w}(t)) \, dt.$$
Now let us consider the case \( v = e_1(x, \lambda) \) and \( w = e_2(x, \lambda) \), where \( e_1(x, \lambda) \) and \( e_2(x, \lambda) \) are defined by one of the formulae (2.1) and \( \lambda \in [\mu, \nu] \). We shall use \( \hat{u} \) for the derivation in \( \lambda \), \( u' \) for the derivation in \( x \), and \( E_i(x, \lambda) \) for \( \hat{e}_i(x, \lambda) \). Since \( d/dx(E_1 \hat{E}_{21} - E_1 \hat{E}_{12}) = d/dx(E_2 \hat{E}_{11} - E_2 \hat{E}_{12}) = -(RE_1, E_2) \), we have

\[
\frac{d}{d\lambda} W[E_1, E_2](\Lambda) = \left\{ (E_2 \hat{E}_{11} - E_2 \hat{E}_{12}) - (E_1 \hat{E}_{21} - E_1 \hat{E}_{12}) \right\}(0, \Lambda) 
= \int_{-\infty}^{+\infty} \left( R(t)E_1(t, \Lambda), E_2(t, \Lambda) \right) dt \neq 0
\]

if for all values of \( t \in \mathbb{R} : E_1(t, \Lambda) = E_2(t, \Lambda) \neq (0, 0) \), i.e. if \( \Lambda \in [\mu, \nu] \) is an eigenvalue of \( \hat{T} \) such that \( W(e_1, e_2)(\Lambda) \neq 0 \).

Set \( e_i(x, \lambda) = \psi_i(a, \lambda) \varphi(x, \lambda) + \{ \rho_i(\lambda) - \varphi_i(a, \lambda) \} \psi(x, \lambda) \). We have

\[
W[e_1, e_2](\lambda) = \psi_1(a, \lambda)\{ \rho_2(\lambda) - \rho_1(\lambda) \} \neq 0
\]

if \( \lambda \in [\mu, \nu] \) and \( \psi_1(a, \lambda) \neq 0 \).

As for Sturm-Liouville operators with Dirichlet and Neumann boundary conditions [7, chap. 13], the spectrum of the operator generated by \( \tau \) on \([0, a]\) with boundary conditions \( u_2(0) = u_2(a) = 0 \) (respectively \( u_1(0) = u_1(a) = 0 \)) is equal to \{ \( \lambda \ | \ \varphi_2(\lambda) = 0 \) \} (respectively \{ \( \lambda \ | \ \psi_1(\lambda) = 0 \) \}). We can also prove that in each maximal interval included in \{ \( \lambda \ | \ \|D(\lambda)\| \leq 2 \) \}, \( \varphi_2(\lambda) \) and \( \psi_1(\lambda) \) have exactly one zero. In particular, there exists an unique \( \kappa \in [\mu, \nu] \) such that \( \psi_1(\kappa, \kappa) = 0 \).

**Case (i).** \( \kappa \) is not an eigenvalue of \( \hat{T} \) and \( \kappa \in [\mu, \nu] \). If \( \lambda \in [\mu, \nu] \), we have

\[
W[E_1, E_2](\lambda) = -\int_A^B (\Delta P(t)e_2(t, \lambda), E_2(t, \lambda)) dt.
\]

The hypothesis

\[
\int_A^B \{ \max (|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)| \} \ dx \leq \pi/2
\]

implies that there are at most two eigenvalues in \([\mu, \nu] \) (Proposition 3.1) and, with the fact that \( e_2 \) and \( E_2 \) are not trivial, that the cosine of the angle between \( e_2 \) and \( E_2 \) is not negative on \([A, B]\). Indeed,

\[
\theta'(x) = \left( (\lambda R(x) - P(x)) \right) \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}, \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}
\]

and

\[
\bar{\theta}'(x) = \left( (\lambda R(x) - P(x) - \Delta P(x)) \right) \begin{pmatrix} \cos \bar{\theta}(x) \\ \sin \bar{\theta}(x) \end{pmatrix}, \begin{pmatrix} \cos \bar{\theta}(x) \\ \sin \bar{\theta}(x) \end{pmatrix},
\]

thus

\[
|\bar{\theta}'(x) - \theta'(x)| \leq \max (|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|
\]

and, for all \( x \) in \([A, B]\),

\[
|\bar{\theta}(x) - \theta(x)| \leq \int_A^x \{ \max (|\mu|, |\nu|)(r_2(s) - r_1(s)) + (p_2(s) - p_1(s)) + |\delta p_2(s)| \} ds \leq \pi/2,
\]
where $\theta$ and $\tilde{\theta}$ are any two determinations of the angular parts of $e_2$ and $E_2$ respectively, such that $\theta(A) = \tilde{\theta}(A)$, and $\lambda \in \{\mu, \nu\}$. Since $\delta p_1 = \delta p_2$ has a constant sign and is not equal almost everywhere to the null function,

$$W[E_1, E_2](\mu) = -\int_{A}^{B} (\Delta P(t)e_2(t, \mu), E_2(t, \mu)) \, dt$$

and

$$W[E_1, E_2](\nu) = -\int_{A}^{B} (\Delta P(t)e_2(t, \nu), E_2(t, \nu)) \, dt$$

are not null and have the opposite sign of $\delta p_2$. Moreover, the function $W[E_1, E_2](\lambda)$ crosses the $\lambda$-axis at $\lambda = \kappa$ and at every eigenvalue. Therefore, there is exactly one eigenvalue in $]\mu, \nu[$.

In order to prove that $W[E_1, E_2](\lambda)$ crosses the $\lambda$-axis at $\lambda = \kappa$, we introduce

$$f_b(x, \lambda) = \psi_1(a, \lambda)\varphi(x, \lambda) + \{\rho_b(\lambda) - \varphi_1(a, \lambda)\} \psi(x, \lambda)$$

and

$$f_c(x, \lambda) = \{\rho_c(\lambda) - \psi_2(a, \lambda)\} \varphi(x, \lambda) + \varphi_2(a, \lambda) \psi(x, \lambda).$$

We suppose that $\{b, c\} = \{1, 2\}$ and $\rho_c(\kappa) \neq \psi_2(a, \kappa)$. Let $F_i(x, \lambda)$ ($i = 1, 2)$ be the corresponding perturbed functions such that $F_1(., \lambda)$ and $f_1(., \lambda)$ are equal on $[B, \infty[$, and $F_2(., \lambda)$ and $f_2(., \lambda)$ are equal on $(-\infty, A]$. It follows that

$$W[f_b, f_c](\lambda) = \{\rho_c(\lambda) - \psi_2(a, \lambda)\} \{\rho_c(\lambda) - \rho_b(\lambda)\},$$

and

$$W[e_1, e_2](\lambda) = \frac{\psi_1(a, \lambda)}{\rho_c(\lambda) - \psi_2(a, \lambda)} W[f_1, f_2](\lambda),$$

and

$$W[E_1, E_2](\lambda) = \frac{\psi_1(a, \lambda)}{\rho_c(\lambda) - \psi_2(a, \lambda)} W[F_1, F_2](\lambda).$$

Near $\kappa$, $W[f_b, f_c](\lambda)$ and $W[F_1, F_2](\lambda)$ are not null, and $(\partial/\partial \lambda) \psi_1(a, \kappa) \neq 0$ (see below); therefore the function $W[E_1, E_2](\lambda)$ crosses the $\lambda$-axis at $\lambda = \kappa$.

Case (ii). $\kappa$ is an eigenvalue. Then $W[F_1, F_2](\lambda)$ and $\psi_1(a, \lambda)$ cross the $\lambda$-axis at $\lambda = \kappa$ and thus $W[E_1, E_2](\lambda)$ is zero at $\lambda = \kappa$ without crossing the $\lambda$-axis. The result follows in the same way as in case (i).

Case (iii). $\kappa \in \{\mu, \nu\}$. Let us introduce

$$g_i(x, \lambda) = \{\rho_i(\lambda) - \psi_2(a, \lambda)\} \varphi(x, \lambda) + \varphi_2(a, \lambda) \psi(x, \lambda) \quad (i = 1, 2),$$

and let $G_i(x, \lambda)$ be the corresponding perturbed functions such that $G_1(., \lambda)$ and $g_1(., \lambda)$ are equal on $[B, \infty[$, and $G_2(., \lambda)$ and $g_2(., \lambda)$ are equal on $(-\infty, A]$. Since

$$W[g_1, g_2](\lambda) = \varphi_2(a, \lambda) \{\rho_1(\lambda) - \rho_2(\lambda)\},$$

it follows that

$$W[E_1, E_2](\lambda) = \frac{\psi_1(a, \lambda)}{\varphi_2(a, \lambda)} W[G_1, G_2](\lambda).$$
Moreover $\varphi_2(a, \kappa) \neq 0$ and $g_1(\cdot, \kappa) = g_2(\cdot, \kappa)$ is not trivial. Hence $W[G_1, G_2](\kappa) \neq 0$ and $W[G_1, G_2](\kappa)$ has the opposite sign of $\delta p_2$. Since

$$\frac{\partial}{\partial \lambda} \varphi_1(a, \lambda) = \int_0^a \{ -(R(t)\psi(t, \lambda), \psi(t, \lambda))\varphi_1(a, \lambda)$$

$$+ (R(t)\psi(t, \lambda), \varphi(t, \lambda))\psi_1(a, \lambda) \} \, dt$$

and

$$\frac{\partial}{\partial \lambda} \varphi_2(a, \lambda) = \int_0^a \{ (R(t)\varphi(t, \lambda), \varphi(t, \lambda))\varphi_2(a, \lambda)$$

$$- (R(t)\varphi(t, \lambda), \varphi(t, \lambda))\varphi_2(a, \lambda) \} \, dt$$

(see [2, lemma 2.1]), we have $(\partial/\partial \lambda)\varphi_2(a, \delta) \neq 0$, where $\delta$ is the unique zero of $\varphi_2(a, \lambda)$ in $[\mu, \nu]$, and $\text{sgn} \{(\partial/\partial \lambda)\varphi_1(a, \kappa)\} = -\text{sgn} \{(\varphi_1(a, \kappa)\} = -\text{sgn} \{(D(\kappa)\} = -\text{sgn} \{(\partial/\partial \lambda)\varphi_2(a, \delta)\}$, and thus $-(\varphi_1(a, \lambda)/\varphi_2(a, \lambda))$ is negative between $\kappa$ and $\lambda$. The results follows as in case (i).

Remarks. 3.3. If $r_1 = r_2$ and $\delta p_1 = \delta p_2$, then Proposition 3.2 provides sufficient conditions for the perturbed operator to have exactly one eigenvalue in each gap, and Proposition 3.1 provides a sufficient condition on $\text{supp}(\Delta P)$ for the perturbed operator to have at most $N + 1$ eigenvalues in each gap ($N \in \mathbb{N}$).

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References


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