# $H_\infty$ Controller Design for Spectral MIMO Models by Convex Optimization[☆]

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**Abstract**

A new method for robust fixed-order $H_\infty$ controller design by convex optimization for multivariable systems is investigated. Linear Time-Invariant Multi-Input Multi-Output (LTI-MIMO) systems represented by a set of complex values in the frequency domain are considered. It is shown that the Generalized Nyquist Stability criterion can be approximated by a set of convex constraints with respect to the parameters of a multivariable linearly parameterized controller in the Nyquist diagram. The diagonal elements of the controller are tuned to satisfy the desired performances, while simultaneously, the off-diagonal elements are designed to decouple the system. Multimodel uncertainty can be directly considered in the proposed approach by increasing the number of constraints. The simulation examples illustrate the effectiveness of the proposed approach.

*Key words:* Convex Optimization, $H_\infty$ control, MIMO systems, spectral models

## 1. Introduction

Most of the industrial plants consist of several interconnected loops, which can be represented by MIMO models. One approach to design multivariable controllers is the classical optimal and robust control techniques applied to a state space representation of these MIMO models. Unfortunately, these

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[1]This research work is financially supported by the Swiss National Science Foundation under Grant No. 200020-107872.

techniques lead to high-order multivariable controllers with a state-space representation. This type of controller structure is not common in industrial plants and their retuning is difficult for control technicians. Several fixed-order controller design methods based on parametric models have been proposed in the literature, e.g. [1, 2, 3]. However, these methods cannot deal with systems containing a time-delay which is very common in industrial processes. Hence, they are rarely used in industry for tuning multivariable PID controllers.

A two-step technique is commonly used in practice instead. In the first step the MIMO system is transformed into a diagonally dominant system using a decoupling precompensator. Once the system is diagonally dominant, SISO techniques are used to design the decoupled controllers for each diagonal element of MIMO system. This strategy is easy to implement and maintain, and is very effective in practice. An example of a two-step approach is given in [4], where first a decoupler is obtained based on the adjoint of the system. Then, a diagonal PID controller is tuned minimizing the integrated absolute error for a step load disturbance for each decoupled system satisfying an upper bound on the sensitivity and complementary sensitivity functions. Many decoupling techniques have been proposed in literature. The classical decoupling methods are based on the eigenvalue decomposition [5] or the singular value decomposition [6]. The minimization of a non-convex function of the weighted off-diagonals of the open-loop system in some given frequencies is considered in [7] to tune a decoupler. An appropriate choice of the weighting function provides a better decoupling around the cross-over frequency.

Since the decoupling step is never perfect, several methods based on a detuning factor are proposed that take into account the coupling effects in the design of SISO controllers. The biggest log modulus tuning (BLT) method proposed in [8] is used to tune individual PI controllers for each decoupled loop using Ziegler-Nichols approach. Then, the proportional and integral terms are multiplied by a detuning factor so that the maximum modulus of the closed-loop transfer function has a specific value.

A decentralized PID controller design method using Gershgorin bands is proposed in [9]. By solving a system of nonlinear equations involving the Gershgorin bands, the decentralized controllers are tuned so that desired gain and phase margins are guaranteed for the diagonal system. It should be noticed that the global stability is not guaranteed because only two crossover frequencies associated to the gain and phase margins are considered for the Gershgorin bands shaping. The coupling effects for a particular loop from all other closed loops are incorporated to a so called effective transfer function, which is used consequently to design decentralized controllers using the single loop tuning techniques [10]. Several simulation examples shows the effectiveness of this approach, however, the stability of the multivariable system cannot be guaranteed.

Although obtaining a parametric model based on physical laws or identification from data is usually too difficult or time consuming, most of the SISO and MIMO controller design methods are based on parametric models. This kind of models need several a priori information, in contrast to spectral models. Spectral models are largely used in practice, however, there is only a few controller design methods based on this type of SISO models, and even fewer for MIMO models. For diagonally dominant or decoupled systems, in [11] a non-convex frequency criterion is defined as the weighted sum of squared error between the desired and computed stability margins considering the Gershgorin bands. Then, this criterion is minimized iteratively using the measured data from some specific closed-loop relay tests. The Gershgorin bands are also used to compute the detuning factor for Z-N tuned controllers based on the calculation of the ultimate gains and ultimate frequencies of each loop using the frequency response of system [12].

The minimization of a weighted difference between a desired diagonal closed-loop frequency response and the real response for some finite frequencies is presented in [13] to attain the decoupling and desired performances via separate non-convex optimizations. Note that the method is only applicable to $2 \times 2$ systems. It should be noticed that these methods do not guarantee the closed-loop stability.

An iterative correlation-based controller (CbT) tuning approach was proposed in [14], where diagonal and off-diagonal elements of the controller transfer function matrix are tuned simultaneously to satisfy desired single-loop closed-loop performance and to decouple the closed-loop outputs. The method is a time-domain data-driven approach using non-convex optimization which does not guarantee the closed-loop stability.

In this work, the method proposed in [15, 16] is extended to deal with LTI-MIMO systems. It is shown that fixed-order linearly parameterized MIMO controllers for MIMO nonparametric spectral models can be computed by convex optimization. The stability of the closed-loop system is guaranteed thanks to the Generalized Nyquist Stability criterion. It should be mentioned that the use of this criterion leads to a non-convex set on the controller parameters. In this paper, a convex approximation of this set is given by a set of convex constraints in the Nyquist diagram based on Gershgorin bands. In this approach, decouplers and decoupled controllers are designed simultaneously by a convex optimization technique. The proposed method can be used for PID controllers as well as for higher order linearly parametrized controllers in discrete or continuous time. The case of unstable open-loop systems can be considered if a stabilizing controller is available.

This paper is organized as follows: In Section 2 a review of the recently proposed method for fixed-order $H_\infty$ controller design for SISO systems in [15] is given. Section 3 introduces the control design methodology for MIMO systems based on the convex constraints in the Nyquist diagram, which guarantees the Generalized Nyquist Stability criterion and single-loop $H_\infty$-SISO performance. This leads to a Semi-Infinite Programming (SIP) problem which is solved using the *scenario approach* in Section 4. Simulation results and comparison with other design methods are given in Section 5. Finally, Section 6 gives some concluding remarks.

## 2. Summary of the approach for SISO systems

In this section, the main idea of fixed-order $H_\infty$ controller design for SISO systems proposed in [15] is reviewed.

A system represented by a set $\mathcal{G}_c$ of $m$ LTI-SISO strictly proper spectral models with bounded infinity norm are considered:

$$\mathcal{G} = \{G_1(j\omega), \dots, G_m(j\omega); \quad \omega \in \mathbb{R}\} \tag{1}$$

The models can be obtained from a set of input/output data via spectral analysis. In the sequel, we consider a continuous-time model $G(j\omega) \in \mathcal{G}$ to explain the basis of the proposed approach. It will be shown that the results can be extended straightforwardly to the case of multiple models and discrete-time systems.

The objective is to design a linearly parameterized controller given by :

$$K(s) = \rho^T \phi(s) \tag{2}$$

where

$$\rho^T = [\rho_1, \rho_2, \dots, \rho_n] \tag{3}$$

$$\phi^T(s) = [\phi_1(s), \phi_2(s), \dots, \phi_n(s)] \tag{4}$$

$n$ is the number of controller parameters and $\phi_i(s), i = 1, \dots n$ are stable transfer functions possibly with poles on the imaginary axis, chosen from a set of orthogonal basis functions. It is clear that PID controllers belong to this set. The main property of this parameterization is that every point on the Nyquist diagram of $L(j\omega, \rho) = K(j\omega)G(j\omega)$ can be written as a linear function of the controller parameters $\rho$:

$$K(j\omega)G(j\omega) = \rho^T \phi(j\omega)G(j\omega) = \rho^T \mathcal{R}(\omega) + j\rho^T \mathcal{I}(\omega) \tag{5}$$

where $\mathcal{R}(\omega)$ and $\mathcal{I}(\omega)$ are the real and imaginary parts of $\phi(j\omega)G(j\omega)$, respectively.

Typically, a standard robust control problem is to design a controller that satisfies $\|W_1\mathcal{S}\|_\infty < 1$ for a set of models, where $W_1(s)$ is the performance weighting filter and $\mathcal{S}$ is the sensitivity function. If the set of models is represented by multiplicative uncertainty, i.e. $\tilde{G}(s) = G(s)[1 + W_2(s)\Delta(s)]$ where $\Delta(s)$ is a stable transfer function with $\|\Delta\|_\infty < 1$, the robust performance condition is given by [17]:

$$|W_1(j\omega)\mathcal{S}(j\omega)| + |W_2(j\omega)\mathcal{T}(j\omega)| < 1 \quad \forall\omega \in \mathbb{R} \tag{6}$$

where

$$\mathcal{S}(j\omega) = [1 + L(j\omega, \rho)]^{-1}$$

is the sensitivity function and

$$\mathcal{T}(j\omega) = L(j\omega, \rho)[1 + L(j\omega, \rho)]^{-1}$$

is the complementary sensitivity function. This constraint is satisfied if and only if there is no intersection in the Nyquist diagram between the performance circle centered at the critical point with a radius of $|W_1(j\omega)|$ and the uncertainty circles centered at $L(j\omega)$ with a radius of $|W_2(j\omega)L(j\omega)|$ at all $\omega$. Equivalently, this condition is satisfied if and only if the circle centered at $L(j\omega, \rho)$ does not intersect the line $d^*(\omega)$, which is tangent to the circle with radius $|W_1(j\omega)|$ and orthogonal to the line between the critical point and $L(j\omega, \rho)$, and is in the side that excludes the critical point (in the right hand side of $d^*(\omega)$ in Fig. 1). This leads to nonconvex robust performance constraints on controller parameters because the line $d^*(\omega)$ depends on the controller parameters.

The basic idea of the proposed approach in [15] is to approximate the nonconvex robust performance constraints in (6) by convex constraints. The line $d^*(\omega)$ can be approximated by the line $d(\omega)$ which is tangent to the performance circle but orthogonal to the line connecting the critical point to $L_d(j\omega)$, which is a desired open-loop transfer function (see Fig. 1). It is clear that if the uncertainty circles are in the right hand side of $d(\omega)$, they have no intersection with the performance circle and the robust performance constraints in (6) are satisfied.

Denoting $x$ and $y$ as the real and imaginary parts of a point on the complex plane, the line $d(\omega)$ is defined as:

$$|W_1(j\omega)[1 + L_d(j\omega)]| - I_m\{L_d(j\omega)\}y - [1 + R_e\{L_d(j\omega)\}](1 + x) = 0 \quad (7)$$

It should be noted that the equation of $d(\omega)$ at each frequency depends only on $W_1(j\omega)$ and $L_d(j\omega)$, which are supposed to be completely known. Therefore, the next linear constraint guarantees that the point $L(j\omega, \rho)$ is in the side of $d(\omega)$ that excludes the critical point:

$$\begin{aligned} &|W_1(j\omega)[1 + L_d(j\omega)]| - I_m\{L_d(j\omega)\}I_m\{L(j\omega, \rho)\}- \\ &\qquad [1 + R_e\{L_d(j\omega)\}][1 + R_e\{L(j\omega, \rho)\}] < 0 \quad \forall\omega \in \mathbb{R} \end{aligned} \quad (8)$$

This linear constraint can be simplified to:

$$|W_1(j\omega)[1 + L_d(j\omega)]| - R_e\{[1 + L_d(-j\omega)][1 + L(j\omega, \rho)]\} < 0 \quad \forall\omega \in \mathbb{R} \quad (9)$$

To assure that all models inside the uncertainty circles satisfy the condition (9), the radius of the performance filter is increased by $|W_2(j\omega)L(j\omega)|$ which gives the following convex constraints with respect to the controller parameter vector $\rho$:

$$\begin{aligned} &|W_1(j\omega)[1 + L_d(j\omega)]| + |W_2(j\omega)L(j\omega, \rho)[1 + L_d(j\omega)]|- \\ &\qquad R_e\{[1 + L_d(-j\omega)][1 + L(j\omega, \rho)]\} < 0 \quad \forall\omega \in \mathbb{R} \end{aligned} \quad (10)$$

These constraints are the convex approximation of the robust performance condition in (6). The conservatism of this approximation depends on the choice of $L_d(j\omega)$. Since $L_d(j\omega)$ is chosen such that it represents some desired control specifications, then it is judicious to minimize a norm of $L - L_d$ under the new convex robust performance constraints. This leads to a Semi-Infinite Programing problem (SIP), an optimization problem with a finite number of variables and an infinite number of constraints:

$$\begin{aligned} &\min_{\rho} \|L(\rho) - L_d\| \\ &\text{Subject to:} \\ &|W_1(j\omega)[1 + L_d(j\omega)]| + |W_2(j\omega)L(j\omega, \rho)[1 + L_d(j\omega)]|- \\ &R_e\{[1 + L_d(-j\omega)][1 + L(j\omega, \rho)]\} < 0 \quad \omega \in \mathbb{R} \end{aligned} \quad (11)$$

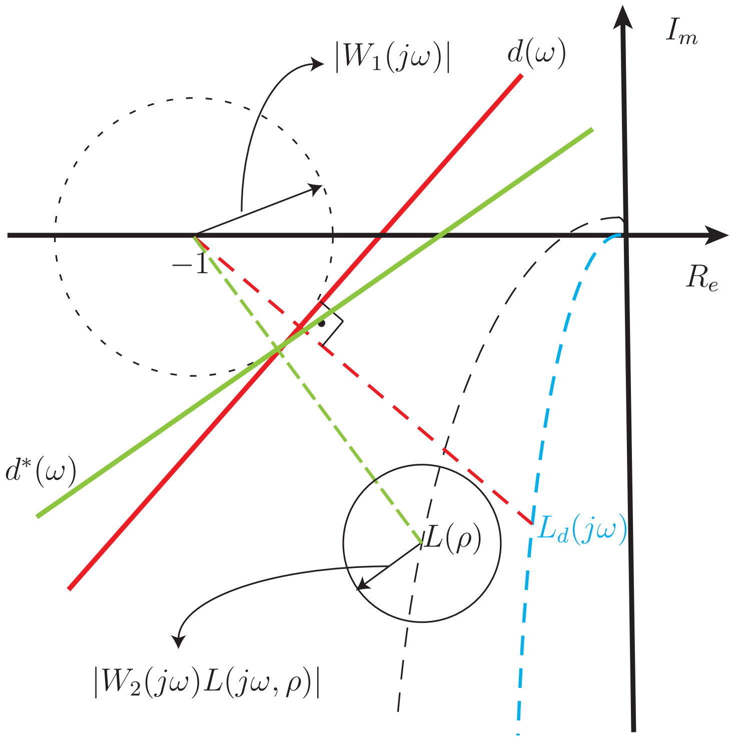


Figure 1: Convex constraints for robust performance in Nyquist diagram

**Remarks:**

- This optimization problem can be extended to the case of multimodel uncertainty. The constraints should be repeated for $m$ models by replacing $L(j\omega,\rho)$ with $L_i(j\omega,\rho) = K(j\omega)G_i(j\omega)$ for $i = 1,\ldots,m$ and the objective function with $\min_{\rho} \sum_{i}^{m} \|L_i(\rho) - L_d\|$.
- It can be shown that the proposed approach can be applied to unstable systems as well. The only condition is that $L_d(s)$ has the same number of unstable poles (with negative real parts) as $G(s)$ and the same poles on the imaginary axis.

## 3. MIMO Controller design in Nyquist diagram

In this section the proposed design method is extended to MIMO systems.

### *3.1. Class of models*

A set $\mathcal{G}$ including $m$ LTI-MIMO strictly proper spectral models with bounded infinity norm are considered:

$$\mathcal{G} = \{\mathbf{G}_1(j\omega), \ldots, \mathbf{G}_m(j\omega); \quad \omega \in \mathbb{R}\} \tag{12}$$

where $\mathbf{G}_i(j\omega)$ is an $n_o \times n_i$ matrix of frequency response functions (FRF) or spectral models with $n_i$ the number of inputs and $n_o$ the number of outputs of the system.

In the sequel, it will be shown that ensuring closed-loop performance and stability for each member of the set $\mathcal{G}$ leads to a set of convex constraints. Therefore, thanks to the properties of convex sets, by repeating the constraints for $m$ models the robust performance and stability for the set $\mathcal{G}$ will be guaranteed. For the sake of simplicity, henceforth, we derive the stability and performance conditions for a single nominal model $\mathbf{G}$. Obviously the results can be applied to the class of model $\mathcal{G}$ by repeating the constraints for every model in the set.

### *3.2. Class of controllers*

Consider the class of multivariable controllers given by an $n_i \times n_o$ matrix $\mathbf{K}(s)$ whose elements $K_{pq}(s)$ for $p = 1,\ldots,n_i$ and $q = 1,\ldots,n_o$ are linearly parameterized. It means that $K_{pq}(s) = \rho_{pq}^T \phi_{pq}(s)$ where $\rho_{pq}^T$ is the vector of parameters for $K_{pq}(s)$ and $\phi_{pq}(s)$ is the vector of stable transfer functions possibly with poles on the imaginary axis, chosen from a set of orthogonal basis functions. Obviously, PID matrix controllers belong to this set, whose non-diagonal elements principally decouple the system and the diagonal elements are designed to achieve some single-loop desired performances. As for the SISO case, the main property of this parameterization is that every component of the matrix $\mathbf{L}(j\omega,\rho) = \mathbf{G}(j\omega)\mathbf{K}(j\omega)$ on the Nyquist diagram, can be written as a linear function of the controller parameters $\rho$:

$$\rho = [\rho_{11}, \ldots, \rho_{1n_i}, \ldots, \rho_{n_o1}, \ldots, \rho_{n_on_i}]$$

*3.3. Stability condition based on Gershgorin Bands*

Let the open-loop system $\mathbf{L}(s)$ have no uncontrollable and/or unobservable unstable modes. Then, the Generalized Nyquist Stability criterion shows that the feedback system will be stable if and only if the net sum of anti-clockwise encirclements of the critical point $(-1+j0)$ by the set of eigenvalues of the matrix $\mathbf{L}(j\omega)$ is equal to the total number of right-half plane poles of $\mathbf{L}(s)$.

The eigenvalues of the matrix $\mathbf{L}(j\omega,\rho)$ at each frequency $\omega$ are nonconvex functions of the controller parameters. A sufficient stability condition can be obtained by approximating the eigenvalues using the Gershgorin bands. Let $\mathbf{L}(j\omega,\rho)$ be the open-loop $n_o \times n_o$ matrix with complex elements $L_{pq}(j\omega,\rho)$. For $q \in \{1,\ldots,n_o\}$ we define

$$r_q(\omega,\rho) = \sum_{p=1,p\neq q}^{n_o} |L_{pq}(j\omega,\rho)| \qquad (13)$$

which is a convex function with respect to the controller parameters. Let $D(L_{qq}(j\omega,\rho), r_q(\omega,\rho))$ be a circle centered at $L_{qq}(j\omega,\rho)$ with radius $r_q(\omega,\rho)$. Such a circle is called a Gershgorin band. Every eigenvalue of $\mathbf{L}(j\omega,\rho)$ lies within at least one of the Gershgorin bands $D(L_{qq}(j\omega,\rho), r_q(\omega,\rho))$ for $q = 1,\ldots,n_o$ [18].

**Proposition 1.** *[19]: Consider that the elements of the $n_o \times n_o$ matrix $\mathbf{L}(j\omega) = \mathbf{G}(j\omega)\mathbf{K}(j\omega)$ satisfy*

$$|r_q(\omega)| < |1 + L_{qq}(j\omega)| \qquad (14)$$

*for $q = 1, \dots, n_o$ and for all $\omega$ on the Nyquist contour, where*

$$r_q(\omega) = \sum_{p=1, p\neq q}^{n_o} |L_{pq}(j\omega)| \tag{15}$$

*Let the $q$th Gershgorin band of $\mathbf{L}(j\omega)$, which is composed of circles centered at $L_{qq}(j\omega)$ with radius $r_q(\omega)$, encircles the critical point $(-1, j0)$, $N_q$ times counterclockwise. Then, the negative feedback system is stable if and only if*

$$\sum_{q=1}^{n_o} N_q = P_0 \tag{16}$$

*where $P_0$ is the number of unstable poles of $\mathbf{L}(s)$.*

Hence, for an open-loop stable system, the closed-loop is stable if the set of the Gershgorin bands of radius $r_q(\omega, \rho)$ of the matrix $\mathbf{L}(j\omega, \rho)$ is strictly at the right hand side of a line passing through the critical point $(-1 + j0)$ for all $\omega$ and for $q = 1, \dots, n_o$. A line $d_q(\omega)$ could be used to divide the complex plane in two half-planes, shown in Fig. 2. The slope of this line can be changed automatically to enlarge the set of admissible controllers if a desired strictly proper open-loop transfer function $L_{Dq}(s)$ is defined for each $q$-th diagonal component. At each frequency $\omega$, the line $d_q(\omega)$ which crosses the critical point $(-1 + j0)$ and is orthogonal to the line connecting the critical point to $L_{Dq}(j\omega)$ is defined. For an open-loop stable system, if all Gershgorin bands for all frequencies are located on the same side of $L_{Dq}(j\omega)$ with respect to the lines $d_q(\omega)$, stability is guaranteed. Therefore an $n_o \times n_o$ desired transfer function matrix $\mathbf{L}_D(j\omega)$ can be defined with $L_{Dq}(j\omega)$ as the $q$-th diagonal element.

The proposed approach can also be applied to unstable systems. The main condition is that $\mathbf{L}_D(j\omega)$ should be a matrix of strictly proper transfer functions and the set of its eigenvalues has to encircle $P_0$ times the critical point $(-1 + j0)$, where $P_0$ is the number of unstable poles of $\mathbf{G}(s)$ (in our approach $K(s)$ has no unstable poles due to its parameterization). This is shown in the following theorem:

**Theorem 1.** *Given the spectral model $\mathbf{G}(j\omega)$, the linearly parameterized controller $\mathbf{K}(s)$ defined in subsection 3.2 stabilizes the closed-loop system if*

$$|r_q(\omega, \rho)| - \frac{R_e\{[1 + L_{Dq}(-j\omega)][1 + L_{qq}(j\omega, \rho)]\}}{|1 + L_{Dq}(j\omega)|} < 0 \qquad \forall\omega \quad \textit{for } q = 1, \dots, n_o \tag{17}$$

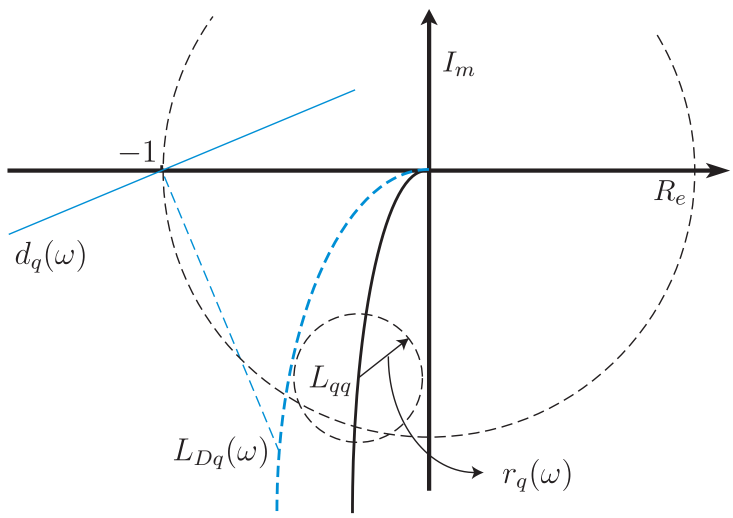


Figure 2: Convex constraints for the Generalized Nyquist Stability criterion in Nyquist diagram

*where the diagonal matrix $\mathbf{L}_D(j\omega)$ is chosen such that the number of counterclockwise encirclements of the critical point by the Nyquist plot of the set of its eigenvalues is equal to the number of unstable poles of $\mathbf{G}(s)$.*

*Proof.* Since the real value of a complex number is less than or equal to its magnitude, we have:

$$R_e\{[1 + L_{Dq}(-j\omega)][1 + L_{qq}(j\omega, \rho)]\} \leq |[1 + L_{Dq}(-j\omega)][1 + L_{qq}(j\omega, \rho)]| \tag{18}$$

Then from (17) we obtain:

$$|r_q(\omega, \rho)| - |1 + L_{qq}(j\omega, \rho)| < 0 \quad \forall\omega \quad \text{for } q = 1, \ldots, n_o \tag{19}$$

which leads directly to (14).

Now we should show that this controller stabilizes the system. From (17), we have:

$$R_e\{[1 + L_{Dq}(-j\omega)][1 + L_{qq}(j\omega, \rho)]\} > 0 \quad \forall\omega \tag{20}$$

or wno$\{[1 + L_{Dq}(-j\omega)][1 + L_{qq}(j\omega, \rho)]\} = 0$, where wno stands for winding number around the origin. It should be mentioned that $L_{qq}(j\omega, \rho)$ is zero for the semicircle with infinity radius of the Nyquist contour so the wno of $1 + L_{qq}(j\omega, \rho)$ depends only on the variation of $s$ on the imaginary axis. On the other hand, as $L_{Dq}(j\omega)$ is a strictly proper transfer function, it goes also to zero for this semicircle. Consequently, the wno of $1 + L_{Dq}(-j\omega)$ is also determined by the variation of $s$ on the imaginary axis. Therefore:

$$\sum_{q=1}^{n_o} \text{wno}[1 + L_{Dq}(j\omega)] = \sum_{q=1}^{n_o} \text{wno}[1 + L_{qq}(j\omega, \rho)] \tag{21}$$

Since $\mathbf{L}_D(j\omega)$ satisfies the Generalized Nyquist criterion, $\mathbf{L}(j\omega)$ will do so as well and all closed-loop systems are stable. □

**Remarks:**

- The results of Theorem 1 are valid even if $L_{qq}(j\omega, \rho)$ has some poles on the imaginary axis, say $\{p_1, p_2, \ldots\}$. In this case $\omega \in \mathbb{R} - \{[p_1 - \epsilon, p_1 + \epsilon], [p_2 - \epsilon, p_2 + \epsilon], \ldots\}$ where $\epsilon$ is a small positive value. The stability is guaranteed if $L_{Dq}(s)$ contains the poles on the imaginary axis of $L_{qq}(s, \rho)$, because they will have the same behavior at the small semicircular detour of the Nyquist contour at these poles.

- According to this theorem, $L_{Dq}(s)$ should contain the unstable poles (as well as the poles on the imaginary axis) of $L_{qq}(s)$. If these poles are unknown (when only the frequency response of the system $\mathbf{G}(j\omega)$ is available), but a stabilizing controller $\mathbf{K}_0(s)$ is available, a reasonable choice for $\mathbf{L}_D(j\omega)$ is $\mathbf{G}(j\omega)\mathbf{K}_0(j\omega)$. In this case $L_{Dq}(j\omega)$ in the above constraints should be replaced by the $q$-th eigenvalue, $\lambda_q(j\omega)$, of $\mathbf{L}_D(j\omega)$.

*3.4. Optimization criterion*

The convex stability constraints shown in (17) add some conservatism to the approach since the location of an eigenvalue is no longer considered at a point but inside the circle $D(L_{qq}(j\omega, \rho), r_q(\omega, \rho))$. To reduce this conservatism, the radius of this circle $r_q(\omega, \rho)$ should be minimized. This is equivalent to minimizing the magnitude of the off-diagonal components of the open-loop transfer function matrix $\mathbf{L}(j\omega, \rho)$. Therefore, it is judicious to minimize the following criterion:

$$J(\rho) = \|\mathbf{L}(\rho) - \mathbf{L}_D\|_2 \tag{22}$$

This way, the off-diagonal elements of $\mathbf{L}(j\omega, \rho)$ will be minimized, which helps to decouple the system. On the other hand, the two norm of $L_{qq} - L_{Dq}$

will be minimized for $q = 1, \ldots, n_o$, which ensures the single-loop closed-loop performances. It means that by one optimization, decoupling controller and decoupled controlled systems are designed simultaneously.

A weighted norm of $\mathbf{L}(\rho) - \mathbf{L}_D$ can be minimized to obtain a controller with more decoupling effect or a better tracking of the desired open-loop in a given frequency range.

## 4. Optimization approach

The optimization problem for single-loop controller design in (11) and the stability constraints for the MIMO system in (17) are defined for all $\omega \in \mathbb{R}$, which is an infinite set. This infinity number of constraints should be satisfied to guarantee the stability and performance conditions. This leads to optimization problems with an infinite number of constraints and a finite number of variables, known as semi-infinite programming (SIP) problem, which is difficult to solve and even NP-hard in many cases.

There exist many numerical methods to deal with general SIP problems, for an overview see [20] and [21]. A practical way to solve this optimization problem is to define $\omega_{\max}$, the frequency above which the gains of all closed-loop transfer functions are close to zero and negligible. For discrete-time systems, this will be the Nyquist frequency (half of the sampling frequency). Then by gridding the interval $[0\ \omega_{\max}]$, a finite set of frequency can be obtained and the constraints can be defined in this finite set. This way, the SIP problem is converted to a semi-definite programming (SDP) problem and can be solved using an SDP solver (e.g. SeDuMi [22]). Since this type of solvers can solve efficiently convex optimization problems with more than hundred thousands of constraints, a very fine grid can be chosen to avoid possible inter grid problems.

An alternative is to use the scenario approach for robust optimization. This approach has been investigated in [23] and [24], where a finite number $N$ of constraints extracted randomly from the infinite set of constraints are used in the optimization problem. The theory shows to what extend the solution satisfies the constraints for another randomly chosen set of frequency points.

Selecting a *violation parameter* $\epsilon \in (0, 1)$ and a *confidence parameter* $\beta \in (0, 1)$, if

$$N \geq \frac{1}{\epsilon}\left(\ln\frac{1}{\beta} + n - 1 + \sqrt{2(n-1)\ln\frac{1}{\beta}}\right) \tag{23}$$

where $n$ is the number of design parameters, then, with probability no smaller than $1-\beta$, the solution satisfies all constraints but at most an $\epsilon$-fraction [25]. $\beta$ plays a key role because selecting $\beta = 0$ leads to $N = \infty$, but practically has marginal importance since its logarithm appears in (23), i.e. $N$ does not grow significantly even with selecting $\beta = 10^{-5}$.

Based on this approach, the SIP problem is transformed to an SDP problem by choosing $N$ independent identically distributed samples $\omega_1, \ldots, \omega_N$ from $\mathbb{R}$. Practically, to obtain a quadratic objective function, the two norm is replaced by $\|\mathbf{L}(\rho) - \mathbf{L}_D\|_2^2$, which is approximated by:

$$\|\mathbf{L}(\rho) - \mathbf{L}_D\|_2^2 \approx \sum_{\omega} \|\mathbf{L}(j\omega, \rho) - \mathbf{L}_D(j\omega)\|_F \tag{24}$$

where $\|\cdot\|_F$ is the Frobenius norm. Thus, the following optimization problem is considered:

$$\begin{aligned}
&\min_{\rho} \sum_{k=1}^{N} \|\mathbf{L}(j\omega_k, \rho) - \mathbf{L}_D(j\omega_k)\|_F \\
&\text{Subject to:} \\
&|r_q(\omega_k, \rho)[1 + L_{Dq}(\omega_k)]| - R_e\{[1 + L_{Dq}(-j\omega_k)\}][1 + L_{qq}(j\omega_k, \rho)]\} < 0 \\
&\qquad \text{for} \quad k = 1, \ldots, N \text{ and } \quad q = 1, \ldots, n_o \\
&|W_{1_q}(j\omega_k)[1 + L_{Dq}(j\omega_k)]| - R_e\{[1 + L_{Dq}(-j\omega_k)][1 + L_{qq}(j\omega_k, \rho)]\} < 0 \\
&\qquad \text{for} \quad k = 1, \ldots, N \text{ and } \quad q = 1, \ldots, n_o
\end{aligned} \tag{25}$$

It is interesting to note that the probability distribution function of the frequency samples has no effect on the bound in (23).

**Remark:** The objective function is convex in $\rho$. If a parametric model of the plant $\mathbf{G}(s)$ and $\mathbf{L}_D(s)$ are available, this convex function can be computed using the state space representation. If only the spectral models are available, this function can be estimated using a finite number of equally spaced frequency samples (not necessarily those used for the constraints). It should be mentioned that a good approximation of the two norm can be obtained if the number of frequency samples is large enough. Increasing the number of frequency samples for estimation of the two norm will not increase the number of constraints and will not complicate the optimization problem.

## 5. Simulation Example

In this section, the proposed algorithm is applied to two industrial processes proposed in literature.

### *5.1. Example 1*

Consider the following process proposed in [10]:

$$\mathbf{G}_1(s) = \begin{bmatrix} \frac{5e^{-3s}}{4s+1} & \frac{2.5e^{-5s}}{15s+1} \\ \frac{-4e^{-6s}}{20s+1} & \frac{e^{-4s}}{5s+1} \end{bmatrix} \tag{26}$$

where the time scale is given in minutes. A continuous-time PI controller should be tuned assuring 3 dB gain margin and $\pi/3$ phase margin. This system is not diagonally dominant and the variable pairings are not evident.

The results are compared with a controller given in [10] where single loop tuning techniques are used to design the decentralized controller using the effective transfer function. This effective transfer function considers the coupling effects for a particular loop from the other closed loop. This results in the following controller:

$$\mathbf{K}_0(s) = \begin{bmatrix} 0.0233\left(1+\frac{1}{4s}\right) & 0 \\ 0 & 0.1094\left(1+\frac{1}{5s}\right) \end{bmatrix} \tag{27}$$

For this example, the number of optimization variables $n$ is 8 and if we choose $\beta = 0.1$ and $\epsilon = 0.1$, we obtain $N = 150$ frequency points for the scenario approach optimization problem. The system is evaluated at $N$ randomly chosen frequency points between 0.01 rad/min and 10 rad/min. The lower limit is greater than 0 because of the integrator. The upper limit is chosen sufficiently large so that the frequency response of the system is negligible at frequencies above the upper limit. The desired open-loop transfer function matrix $\mathbf{L}_D(s)$ is chosen as simple integrators at the diagonal elements with a bandwidth similar to those obtained with the controller proposed in [10]:

$$\mathbf{L}_D(s) = \begin{bmatrix} \frac{1}{30s} & 0 \\ 0 & \frac{1}{30s} \end{bmatrix} \tag{28}$$

which satisfies the specified gain and phase margins.

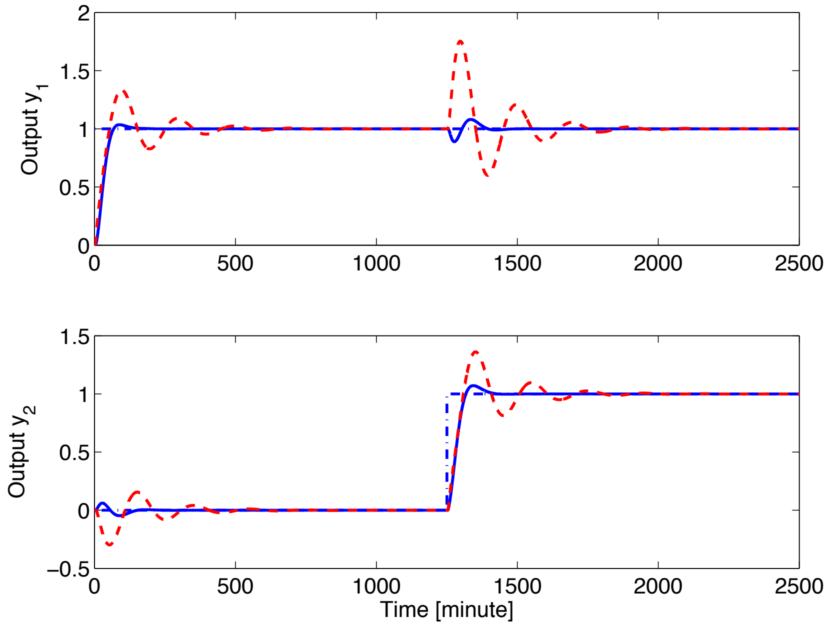


Figure 3: Two output responses: Reference signal (blue, dash-dot), controller proposed in [10] (red dashed) and proposed controller (blue solid).

Only stability constraints are considered on the optimization problem given in (25), which results in the following controller:

$$\mathbf{K}_1(s) = \begin{bmatrix} \frac{0.002667s+0.002338}{s} & \frac{0.003439s-0.005726}{s} \\ \frac{-0.0004078s+0.008943}{s} & \frac{0.05531s+0.01166}{s} \end{bmatrix} \tag{29}$$

Figure 3 shows that the proposed method decouples the system almost perfectly, which is not the case for $\mathbf{K}_0(s)$. It should be noted that the controller proposed in [10] is a decentralized controller while that proposed in this paper is centralized. The complexity of the controller explains the better performances obtained with the proposed controller. However, even that the proposed method offers the possibility to design more complex controllers, the method remains simple and intuitive.

One of the most important advantages of the proposed approach is that systems with multi-model uncertainty can directly be considered. A second system with 100% higher values for the gains, time constants and time delays than those of the previous system $\mathbf{G}_1(s)$ is defined:

$$\mathbf{G}_2(s) = \begin{bmatrix} \frac{10e^{-6s}}{8s+1} & \frac{5e^{-10s}}{30s+1} \\ \\ \frac{-8e^{-12s}}{40s+1} & \frac{2e^{-8s}}{10s+1} \end{bmatrix} \tag{30}$$

The following multiplicative uncertainty filters are defined for the diagonal elements of both systems, $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ by:

$$W_{2_q}(s) = 0.5\frac{2s+1}{s+1} \quad \text{for } q = 1,2 \tag{31}$$

A stabilizing PI MIMO controller is tuned to satisfy the robust performance condition in (6) for the diagonal elements of both systems, where the performance filter for both systems is given by $W_{1_q}(s) = 0.5$ for $q = 1, 2$.

The optimization problem proposed in (25) is solved by repeating the stability and robust performance constraints for $\mathbf{G}_2(s)$. This results in the following controller:

$$\mathbf{K}_2(s) = \begin{bmatrix} \frac{0.001851s+0.001348}{s} & \frac{0.002225s-0.003084}{s} \\ \\ \frac{-0.0005015s+0.004521}{s} & \frac{0.03111s+0.006742}{s} \end{bmatrix} \tag{32}$$

This controller is stabilizing and satisfies the required $H_\infty$ constraints for both systems. Figure 4 shows that, contrarily to $\mathbf{K}_1(s)$, controller $\mathbf{K}_2(s)$ decouples both systems. This was expected because $\mathbf{K}_1(s)$ was not designed with this porpoise. It should be noted that $\mathbf{K}_0(s)$ does not even stabilize $\mathbf{G}_2(s)$.

### *5.2. Example 2*

In this example, the proposed algorithm is applied to a multivariable gas turbine engine and the results are compared with those of two other data-driven controller design approaches for multivariable systems in [26, 14]. The objective is to tune a multivariable PI controller for an LV100 gas turbine engine to follow the reference model $\mathbf{M}_D$. The plant is represented by a continuous-time state-space model with five states, two inputs and two outputs. The model is discretized using Tustin's approximation with $T_s = 0.1s$ as sampling period. Each experiment is performed with a measurement noise that is generated as a zero-mean, stationary white Gaussian sequence with variance $0.0025I$.

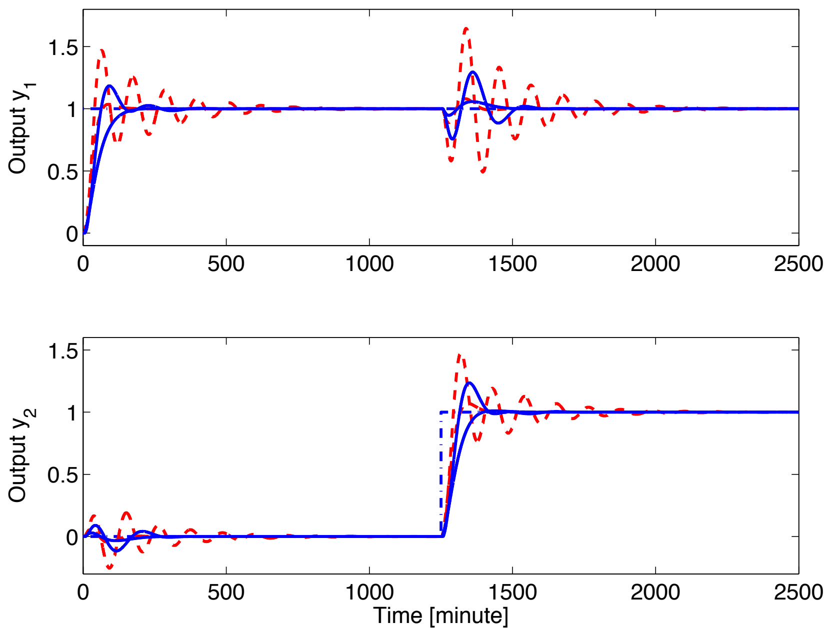


Figure 4: Two output responses: Reference signal (blue, dash-dot), $\mathbf{K}_1(s)$ (red, dashed) and $\mathbf{K}_2(s)$ (blue, solid).

The given reference model is:

$$\mathbf{M}_D(z) = \begin{bmatrix} M_{d_1} & 0 \\ 0 & M_{d_2} \end{bmatrix} = \begin{bmatrix} \frac{0.4}{z-0.6} & 0 \\ 0 & \frac{0.4}{z-0.6} \end{bmatrix} \tag{33}$$

which is used to define our desired open-loop transfer function:

$$\begin{aligned} \mathbf{L}_D(z) &= \begin{bmatrix} \frac{M_{d_1}}{1-M_{d_1}} & 0 \\ 0 & \frac{M_{d_2}}{1-M_{d_2}} \end{bmatrix} \\ &= \begin{bmatrix} L_{d_1} & 0 \\ 0 & L_{d_2} \end{bmatrix} = \begin{bmatrix} \frac{0.4}{z-1} & 0 \\ 0 & \frac{0.4}{z-1} \end{bmatrix} \end{aligned} \tag{34}$$

An experiment is realized using the simulation conditions proposed in [26]. The results are compared with the CbT controller given in [14]:

$$\mathbf{K}_{CbT}(z) = \begin{bmatrix} \frac{0.3636z-0.09866}{z-1} & \frac{0.3653z-0.2691}{z-1} \\ \\ \frac{18.69z-18.16}{z-1} & \frac{-3.453z+2.652}{z-1} \end{bmatrix} \tag{35}$$

and the IFT controller provided in [26]:

$$\mathbf{K}_{IFT}(z) = \begin{bmatrix} \frac{0.248z-0.03}{z-1} & \frac{0.38z-0.199}{z-1} \\ \frac{16.47z-15.91}{z-1} & \frac{0.063z+0.054}{z-1} \end{bmatrix} \tag{36}$$

The sum of squared output errors (SSOE) is used for comparison of different controllers. This criterion is defined as:

$$SSOE = \frac{1}{M}\sum_{t=1}^{M} \varepsilon_{oe}^T(t)\varepsilon_{oe}(t) \tag{37}$$

where M=151 is the data length and $\varepsilon_{oe}$ the difference between the desired and the obtained outputs.

*5.2.1. Model-based design*

As in the previous example, the number of optimization variables $n$ is 8, which gives $N = 150$ frequency points for the same values of $\beta$ and $\epsilon$. These frequency points are selected randomly between $\omega_{\max}/N$ and $\omega_{\max}$ rad/s. The lower limit is greater than zero because of the integrator in the controller and $\omega_{\max}$ is chosen equal to the Nyquist frequency. The optimization problem in (25) without the $H_\infty$ constraints are used. The result of the optimization algorithm is :

$$\mathbf{K}_0(z) = \begin{bmatrix} \frac{0.3779z-0.1098}{z-1} & \frac{0.3584z-0.2574}{z-1} \\ \frac{19.57z-19.02}{z-1} & \frac{-3.183z+2.262}{z-1} \end{bmatrix} \tag{38}$$

with an SSOE equal to 0.0047. Figure 5 shows the experiment without noise where an almost perfect decoupling can be observed.

*5.2.2. Data-driven design*

To be fair in this comparison, a spectral model is identified based on the same rectangular reference signal used by the methods previously mentioned. The Empirical Transfer Function Estimate (ETFE) method is used to identify the frequency function models based on the input/output measurements for the rectangular reference signal applied to each input while the other is not excited. As the input spectrum is low at high frequencies and as the signal to noise ratio is very low, the ETFE model is evaluated at $N = 150$ random frequency points between $1/N$ and $\omega_{\max} = 1$ rad/s for the optimization

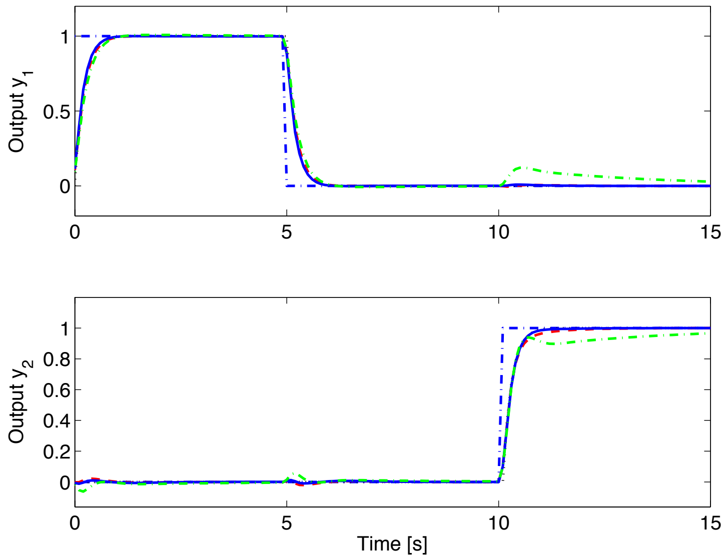


Figure 5: Two output responses: Reference signal (blue, dash-dot), Reference model (black dotted), CbT (red dashed), IFT (green dash-dot) and proposed controller using the model (blue solid)

problem (this range of frequency defines the first lobe of the input spectrum). The resulting PI MIMO controller is :

$$\mathbf{K}(z) = \begin{bmatrix} \frac{0.3359z-0.05477}{z-1} & \frac{0.3624z-0.2615}{z-1} \\ \\ \frac{20.42z-19.79}{z-1} & \frac{-3.308z+2.348}{z-1} \end{bmatrix} \tag{39}$$

Figure 6 shows that the system's outputs using the proposed controller are very close to the desired response except for the effect of noise. In addition, the closed-loop system is nearly fully diagonalized. The observed SSOE with the proposed controller is 0.0048, while those with the CbT and IFT controllers are 0.0050 and 0.0082, respectively. Even if IFT method has a noise-rejection objective function that could be advantageous in a noisy environment, the results are not so satisfactory because it is not able to fully decouple the system, while the other methods do. This is more perceptible in Figure 7, where an experiment without noise is shown. At the instants 0s and 5s on $y_2$ and at instant 10s on $y_1$, it is visible that the decoupling of the IFT controller is not as good as that proposed by the other approaches.

It should be noticed that global stability is guaranteed thanks to the Gershgorin bands considered as convex constraints in this approach, whereas

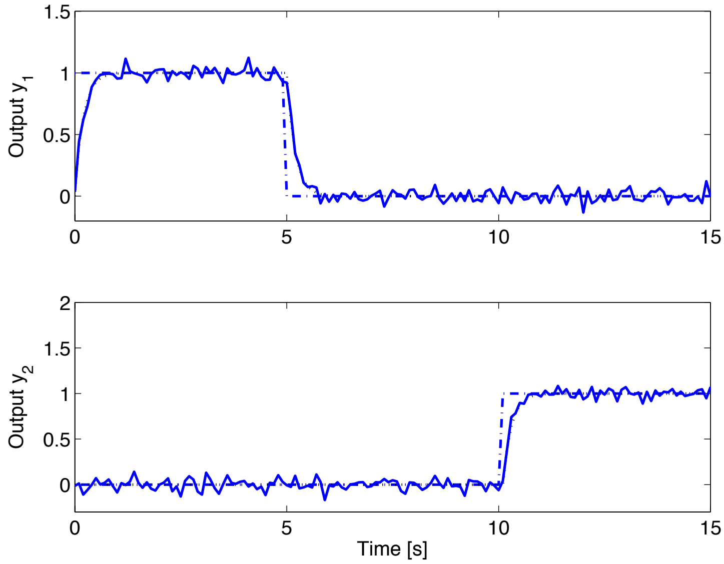


Figure 6: Two output responses: Reference signal (blue, dash-dot), Reference model (black dotted) and proposed controller (blue solided)

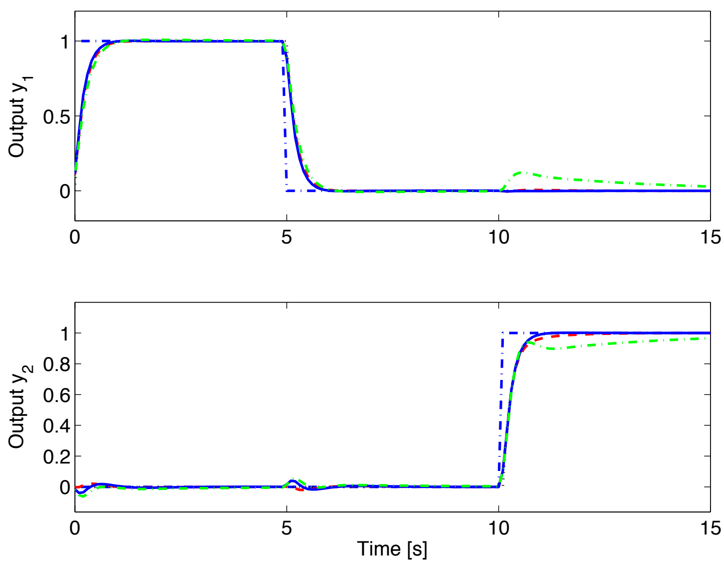


Figure 7: Two outputs response: Reference signal (blue, dash-dot), Reference model (black dotted), CbT (red dashed), IFT (green dash-dot) and proposed controller (blue solided)

the other data-driven approaches do not ensure the stability of the closed-loop system. The other advantage of the proposed approach is in terms of experimental cost. The CbT and IFT methods are iterative methods which are experimentally expensive. The IFT controller is designed after 6 iterations with a total of 450 seconds of experimentation and CbT is obtained after 8 iterations (120 seconds), while the proposed method is designed based on an experiment of 20 seconds. Moreover, the computational complexity of this approach is very low. For this particular example, the Matlab code, running on a Mac Intel 3GHz Xeon 8 cores takes only 5.3 seconds.

## 6. Conclusions

A new decoupling fixed-order MIMO controller design method in the Nyquist diagram for spectral MIMO models has been proposed in this paper. The method is based on an approximation of the Generalized Nyquist Stability criterion that leads to convex constraints with respect to linearly parameterized controllers.

Even if the performance and stability conditions are defined for an infinite number of frequency points, a randomized solution for SIP problem is used that needs a finite number of frequency points. This solution, satisfies with a desired probability all the constraints exept for a defined fraction of constraints. The controller is linearly parameterized and its denominator should be fixed a priori. However, this restriction ensures the stability of the controller and makes no problem for PID controller design.

The advantages of this approach are summarized below:

1. Only the frequency response of the system is needed and no parametric model is required. The method can be considered as "data-driven" because the frequency response of the system can be obtained directly by discrete Fourier transform from a set of periodic data. Of course, when a parametric model is given, the method can also be applied.
2. Simultaneously, the diagonal elements of the controller are tuned to satisfy some desired performances, while the off-diagonal elements are tuned to decouple the system.
3. Multimodel uncertainty can be handled easily by increasing the number of constraints. Most of the mentioned classical frequency-domain approaches cannot deal with this type of uncertainty.
4. If a stabilizing initial controller is known, unstable systems can also be considered with this approach.

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