Generalities and studied problem

Traditionally, a musical glass is shaved just over the stem to obtain the desired pitch. Can the physical cause of this traditional method be identified? In preambule, some vibrational modes of a glass and its frequency spectrum are observed.

![Figure 1: Modal forms and spectrum of a vibrating glass](image)

It is commonly admitted that the first circonfenential mode affects the first partials. The number of harmonics when the glass is excited (Figure 1(c)) and the fast disappearance of the higher orders when the glass is no longer excited are noteworthy. This spectrum cannot be fully explained as of today.

Chosen approach

In the case of a glass of very simple geometry, A.P. French [1] gives in 1983 an expression of the eigenfrequencies that puts the way to considering the tuning problem. In fact, if \( n \) is the order of the circonfenential mode \( (n \geq 2) \) and \( m \) the order of a flexion mode of a generatrix of the cylindrical glass, the frequencies are described as

\[
f_{mn} = K h \sqrt{(n^2 - 1)^2 + (\beta_m b)^4} \left( 1 + \frac{4}{n^2} \right) \tag{1}
\]

where \( \beta_m \approx m - \frac{1}{2} \) (very roughly for \( m = 1 \)) and \( b = \frac{R}{L} \) with \( R \) the radius of the cylinder, \( L \) its height and \( h \) the thickness of its walls. Rewriting

\[
f_{mn} \approx K h \sqrt{(n^2 - 1)^2 + (\beta_m b^4)(1 - \frac{1}{n^2})} \tag{2}
\]

the contribution of the circonfenential index \( n \) and of the flexion index \( m \) are decoupled when \( \frac{1}{n} \) is neglected before 1 (also very roughly for \( n \geq 2 \)). One would therefore obtain \( f_{mn} \approx \sqrt{f_n^2 + f_m^2} \) and actually \( f_m \) is nearly the expression of the frequency of a clamped-free beam in flexion. Moreover, the tapering of the glass’ bottom seems to affect the circonfenential modes \( (f_n^2) \) very little, on the contrary for \( f_m^2 \). It seems therefore natural to consider the tuning problem via the study of the influence of material removal on a generatrix of the glass, assimilated to a clamped-free beam. The considered beam is made of glass (\( E = 6.15 \times 10^10 \) Pa, \( \rho = 2880 \text{kg} \cdot \text{m}^{-3} \)), has a length of 5cm and a thickness of 1.5mm.

Analytical method of small perturbations

A simple situation enlightens the flexion equation of a non-uniform beam

\[
\frac{\partial^4 \zeta}{\partial y^4} + \frac{2 \partial I}{I} \frac{\partial \zeta}{\partial y} + \frac{1}{I} \frac{\partial^2 I}{\partial y^2} \frac{\partial^2 \zeta}{\partial y^2} - \frac{\rho S \omega^2}{EI} \zeta(y) = 0 \tag{3}
\]

Let it be a beam of length \( L \), thickness \( h \), rectangular section \( S \), density \( \rho \), Young’s modulus \( E \), gyration radius \( a \) and moment \( I \). If the thickness varies linearly along \( Oy \), one has \( h(y) = h(0)(1+\mu y) \) or \( h(y) = h(L)(1+\mu y) \) with \( \mu L \ll 1 \). Noting \( h_0 = h(0) \) and \( h_L = h(L) \), the expressions of the first order are: \( S(y) = S_0(1+\mu y), a^2(y) = a_0^2(1+2\mu y), I(y) = I_0(1+3\mu y) \) and \( \gamma(y) = \gamma_0(1-2\mu y) \) with \( S_0 = h_0 t_x, a_0^2 = \frac{b_0^2}{12}, I_0 = \frac{b_0^4 S_0}{12}, \gamma_0 = \frac{C_0}{a_0^2}, C_0 = \frac{\rho S_0}{E} \), from which one obtains

\[
\frac{\partial^4 \zeta}{\partial y^4} + 6 \mu \frac{\partial^3 \zeta}{\partial y^3} - \gamma^4(y)\zeta(y) = 0 \tag{4}
\]

The goals are the eigenfrequencies and modes. The modes \( \zeta_m(y) \), searched in the base of the modes \( \zeta_0(y) \) of the beam of uniform thickness \( h_0 \) (or \( h_L \)) are written \( \zeta_m(y) = \sum_{k=1}^{\infty} b_{mk} \zeta_0(y) \). \( \zeta_0(y) \) outweighs the rest when \( b_{mn} \gg b_{mk} \forall m \neq k \). The awaited frequencies (near the frequencies of the uniform case) are written \( f_m = f_{m0}(1+\theta_m) \) with \( \theta_m \ll 1 \), or \( \omega_m^2 = \omega_{m0}^2(1+2\theta_m) \).

The resolution of the first order being the goal, using the above approximations, \( \frac{\partial^3 \zeta_{mn}}{\partial y^3} = C_0 \omega_{m0}^2 \zeta_0 \) and \( \frac{\partial^4 \zeta_{mn}}{\partial y^4} = C_0 \omega_{m0}^2 \zeta_0 \), one obtains

\[
b_{mn}(-6\mu \frac{\partial^4 \zeta_{mn}}{\partial y^4} - 2\gamma_0 \omega_{m0}^2 \zeta_0 + 2\theta_m C_0 \omega_{m0}^2 \zeta_0) = \sum_{k \neq m} b_{mk} C_0 (a_0^2 - \omega_{m0}^2) \zeta_0 \tag{5}
\]

Since \( \theta_m \) only appears in the first term, a way to isolate it only would give access to \( \theta_m \). In fact, the projection of the total expression on mode \( \zeta_{0j}(y) \) with \( j \neq k \) and particularly for \( j = m \) gives

\[
\theta_m = \left[ 2 \int_0^L \gamma y \zeta_{m0}^2 dy - \frac{3}{\zeta_{m0}^2} \left( \frac{\partial \zeta_{m0}}{\partial y} \right)^2 \right] \frac{\mu}{\gamma_0} = K \frac{\mu}{L} \frac{L}{L} \tag{6}
\]
where the modes \( \zeta_{n0}(y) \) and their derivatives are analytically available [2]. In the case of material removal at the bottom of the glass, the initial situation uses a uniform thickness of \( h_0 \). The eigenfrequency depending directly on \( h \), it appears \( f_m \approx f_{n0}(1 + K_m \frac{h}{L})^{1/2} \). Taking the boundary conditions into account, integration by parts gives

\[
\frac{\partial^3 \zeta}{\partial y^3} + 6\mu \frac{\partial \zeta}{\partial y^2} - 7\gamma_0 (1 - 2\mu y) \zeta(y) = 0 \quad (7)
\]

with an excitation source in \( y_s \). The solution satisfies \( \int_0^L \nu(y) \left[ \frac{\partial^3 \zeta}{\partial y^3} + 6\mu \frac{\partial \zeta}{\partial y^2} - 7\gamma_0 (1 - 2\mu y) \zeta(y) \right] dy = 0 \). The resonance frequencies of the beam are obtained by analytically solving this differential equation.

The flexion equation is

\[
\frac{\partial^2 \zeta}{\partial y^2} + \sigma_0 (1 - 2\mu y) \zeta = \delta(y - y_s) \quad (8)
\]

with an excitation source in \( y_s \). The solution satisfies \( \int_0^L \nu(y) \left[ \frac{\partial^2 \zeta}{\partial y^2} + \sigma_0 (1 - 2\mu y) \zeta(y) \right] dy = 0 \). The eigenfrequency depends analytically with the small perturbation method (see Figure 2).

The flexion modes are preponderant in the tuning process. The authors wish to thank Mrs. I. Emge and Mr. J.-F. Zürcher for motivating them to work on the problematic of the tuning of musical glasses.

**Discussion**

Keeping in mind that the eigenfrequency variation due to the behaviour of the glass generatrix is only a contribution to the eigenfrequency variation of the whole glass, it can nevertheless be stated that the grinding process results in a frequency reduction. Figure 2 shows that a material removal of at least 3% is necessary to reduce the frequency by a half tone (approx. 6% relative variation).

Tradition states that the frequency can be raised by more than one octave by thinning the walls near the top of the glass (here this would mean a negative \( \mu \) with an initial thickness of \( h_0 \)). This can not however be verified with this model, as it would provide only a very slight increase.

It is interesting to compare these frequency variations to the experimental values published in [3], which take the whole glass into account. The results obtained hint that the flexion modes are preponderant in the tuning process.

**References**


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