An LMI formulation of fixed-order H_∞ and H_2 controller design for discrete-time systems with polytopic uncertainty

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***Abstract*— In this paper, a new approach to fixed-order H_∞ and H_2 output feedback control of MIMO discrete-time systems with polytopic uncertainty is proposed. The main idea of this approach is based on the definition of SPR-pair matrices and the use of some instrumental matrices which operates as a tool to overcome the original non-convexity of fixed-order controller design. Then, stability condition as well as H_∞ and H_2 performance constraints are presented by a set of linear matrix inequalities with linearly parameter dependent Lyapunov matrices. Iterative algorithm for update on the instrumental matrices monotonically converges to a suboptimal solution. Simulation results show the effectiveness of the proposed approach.**

I. INTRODUCTION

Recently, linear matrix inequalities (LMIs) emerge as a useful tool to deal with a wide variety of optimization and control problems. Many control design problems can be easily expressed in terms of a set of LMIs which are readily solved using existing standard LMI solvers.

Low-, fixed-order dynamic output-feedback control design of polytopic systems using LMI conditions is a challenging issue in the community of robust control theory and it has attracted considerable attention since the last decade. The problem is originally non-convex in the space of controller parameters and it cannot be solved in polynomial time [1].

Recently, several researchers have developed inner convex approximations of the non-convex set of all fixed-order stabilizing controllers in terms of LMIs for systems with polytopic uncertainty in the polynomial framework [2], [3]. The results have been extended to fixed-order H_∞ controller design in [4], [5]. An LMI formulation of fixed-structure H_2 controller design for SISO transfer functions is also given in [6]. All these approaches are based on the positivity of polynomials and Strictly Positive Realness (SPRness) of transfer functions. The quality of these approaches depends on the choice of a so-called central polynomial. These approaches are limited to systems with polynomially polytopic uncertainty; therefore, they cannot consider state-space polytopic uncertainty which has more general structure than polynomially polytopic uncertainty.

This paper is a continuation of our previous research work in [7], [8] where the problem of fixed-order H_∞ and H_2 control synthesis of continuous-time polytopic systems in the state space framework has been considered by introducing a new concept of SPR-pair matrices. The SPR-pair matrices play a key role as instrumental matrices to convexify the originally non-convex problem of fixed-order controller design by decoupling some unknown matrices.

In this paper, discrete-time systems with polytopic uncertainty are handled by a similar idea. However, the extension of results to discrete-time case is not straightforward specially for fixed-order H_∞ controller design. In Bounded Real Lemma of discrete-time systems, the product of unknown Lyapunov and state matrices appears not only in the stability block (A^TPA-P) but also in other blocks (A^TPB/APC^T). Therefore, in comparison to the continuos-time case in [7], the SPR-pair matrices have to decouple all these unknown matrices.

For both fixed-order H_∞ and H_2 controller design, sufficient conditions in terms of LMI optimization problems are given in this paper. Moreover, an iterative procedure for update on the instrumental matrices is proposed that monotonically converges to a suboptimal solution. This property is one of the important features of the proposed approach in this paper.

The organization of the paper is as follows: Problem statement, preliminaries, basic idea and simulation examples are presented in next section. LMI conditions for fixed-order H_∞ and H_2 controller design are developed in Sections III and IV, respectively. Finally, conclusions are provided in Section V.

The notation used throughout the paper is standard. I is an identity matrix of appropriate dimension. $\star$ indicates symmetric blocks. $P>0$ and $P<0$ means that the matrix P is positive-definite and negative-definite, respectively.

II. FIXED-ORDER STABILIZING CONTROLLERS

A. Problem statement

Consider a dynamical discrete-time system given by:

$$\begin{aligned} x_g(k+1) &= A_g x_g(k) + B_g u(k) + B_w w(k) \\ z(k) &= C_z x_g(k) + D_{zu} u(k) + D_{zw} w(k) \\ y(k) &= C_g x_g(k) + D_w w(k) \end{aligned} \qquad (1)$$

where $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^{n_o}$, and $z \in \mathbb{R}^s$ are the state, the control input, the exogenous input, the measured output, and the controlled output vector, respectively. The real matrices A_g, B_g, B_w, C_z, C_g, D_{zu}, D_{zw}, and D_w are of appropriate dimensions. It is assumed that the matrices A_g and B_g belong to a polytopic region as

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follows:

$$A_g(\lambda) = \sum_{i=1}^{q} \lambda_i A_{g_i} \qquad B_g(\lambda) = \sum_{i=1}^{q} \lambda_i B_{g_i}; \tag{2}$$

where $\lambda = [\lambda_1 \cdots \lambda_q]^T \in \Lambda$,

$$\Lambda = \left\{ \lambda \middle| \quad \sum_{i=1}^{q} \lambda_i = 1, \quad \lambda_i \geq 0; \quad i = 1, \ldots, q \right\} \tag{3}$$

and

$$H_i(z) = \left[\begin{array}{c|c} A_{g_i} & \begin{bmatrix} B_{g_i} & B_w \end{bmatrix} \\ \hline \begin{bmatrix} C_z \\ C_g \end{bmatrix} & \begin{bmatrix} D_{zu} & D_{zw} \\ 0 & D_w \end{bmatrix} \end{array} \right] \tag{4}$$

is the transfer matrix of each vertex of the polytope. This polytopic system contains a variety of parametric uncertainties such as multiple models and interval uncertainty. Note that if the matrix C_g has polytopic uncertainty and the matrix B_g is fixed, very similar results are obtained.

The problem addressed in this section is to present LMI conditions for fixed-order output feedback stabilizing controller design of the polytopic system in (1) and (2). The controller is represented by:

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c y(k) \\ u(k) &= C_c x_c(k) + D_c y(k) \end{aligned} \tag{5}$$

where $A_c \in \mathbb{R}^{m \times m}$ and B_c, C_c, and D_c are of appropriate dimensions. Then, the state space representation of the closed-loop system H_{zw} is given by:

$$\begin{aligned} x(k+1) &= A(\lambda)x(k) + B(\lambda)w(k) \\ z(k) &= Cx(k) + Dw(k) \end{aligned} \tag{6}$$

where

$$\begin{aligned} A(\lambda) &= \begin{bmatrix} A_g(\lambda) + B_g(\lambda) D_c C_g & B_g(\lambda) C_c \\ B_c C_g & A_c \end{bmatrix} \\ B(\lambda) &= \begin{bmatrix} B_w + B_g(\lambda) D_c D_w \\ B_c D_w \end{bmatrix} \\ C &= [C_z + D_{zu} D_c C_g \quad D_{zu} C_c] \\ D &= D_{zw} + D_{zu} D_c D_w \end{aligned} \tag{7}$$

The matrix $A(\lambda)$ is called robustly stable if the magnitude of all its eigenvalues is less than one for all $\lambda \in \Lambda$.

B. Basic idea

The main idea behind the polynomial-based approaches [2], [3] for the fixed order controller design of SISO polytopic systems with rational transfer function representations is based on the strictly positive realness (SPRness) of some transfer functions. These transfer functions are the ratio of the closed-loop characteristic polynomial at the each vertex to a given stable polynomial called the central polynomial. The control performance as well as the conservatism of the approach depends on the choice of the central polynomial. In this paper, the similar idea is used to propose some LMI conditions for fixed-order controller design of discrete-time systems with polytopic uncertainty in their state space representation. The following lemma, definition and theorem provide the basic concepts of this paper concerning the fixed-order stabilizing controller synthesis.

Lemma 1: [5] An SPR transfer matrix $H(z)$ and its inverse $H^{-1}(z)$ are SPR with a common Lyapunov matrix P.

Now, consider two transfer matrices $H(z)$ and its inverse as follows:

$$H(z) = \left[\begin{array}{c|c} A & B \\ \hline C & I \end{array} \right], \quad H^{-1}(z) = \left[\begin{array}{c|c} A - BC & B \\ \hline -C & I \end{array} \right] \tag{8}$$

Based on the Kalman-Yakubovic-Popov (KYP) lemma [10], [11], the SPRness of $H(z)$ and $H^{-1}(z)$ leads to the following equivalent inequalities:

$$\begin{bmatrix} A^T P A - P & \star \\ B^T P A - C & B^T P B - 2I \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} (A - BC)^T P (A - BC) - P & \star \\ B^T P (A - BC) + C & B^T P B - 2I \end{bmatrix} < 0 \tag{10}$$

Remark: The matrices A and $A - BC$ are both stable with a common Lyapunov matrix P.

Definition 1: Two matrices M and A in $\mathbb{R}^{n \times n}$ are called *SPR-pair* matrices if

$$H(z) = \left[\begin{array}{c|c} M & I \\ \hline M - A & I \end{array} \right] \tag{11}$$

is SPR.

By applying Lemma 1, it is obvious that if $H(z)$ in (11) is SPR, $H^{-1}(z)$ with the following state space realization is also SPR.

$$H^{-1}(z) = \left[\begin{array}{c|c} A & I \\ \hline A - M & I \end{array} \right] \tag{12}$$

Therefore, if M and A are SPR-pair, then A and M are also SPR-pair (commutative property) and they are both stable with a common Lyapunov matrix. As a result, the following inequalities are equivalent:

$$\begin{aligned} \begin{bmatrix} M^T P M - P & \star \\ PM - M + A & P - 2I \end{bmatrix} < 0 \\ \begin{bmatrix} A^T P A - P & \star \\ PA - A + M & P - 2I \end{bmatrix} < 0 \end{aligned} \tag{13}$$

The following theorem proposes a new set of LMIs for fixed-order stabilizing controller design of the discrete-time systems with polytopic uncertainty defined in (1) and (2).

Theorem 1: The fixed-order controller defined in (5) stabilizes the discrete-time polytopic system in (1) and (2) if there exist a stable matrix M and a non-singular matrix T such that M and $T^{-1} A_i T$ are SPR-pair for $i = 1, \ldots, q$, where A_i is the closed-loop state matrix of the i-th vertex defined by:

$$A_i = \begin{bmatrix} A_{g_i} + B_{g_i} D_c C_g & B_{g_i} C_c \\ B_c C_g & A_c \end{bmatrix} \tag{14}$$

Therefore, a convex set of stabilizing controllers can be given using the KYP lemma by the following set of LMIs:

$$\begin{bmatrix} M^T P_i M - P_i & \star \\ P_i M - M + T^{-1} A_i T & P_i - 2I \end{bmatrix} < 0 \tag{15}$$

for $i = 1, \ldots, q$.

The above inequalities are LMIs with respect to the controller parameters (A_c, B_c, C_c, D_c) and symmetric Lyapunov matrices P_i for $i = 1, \ldots, q$.

Proof: Since M makes an SPR-pair with $T^{-1}A_iT$ in (15), both matrices are stable with the Lyapunov matrix P_i. By convex combination of (15) for all vertices, one can obtain:

$$\begin{bmatrix} M^TP(\lambda)M - P(\lambda) & \star \\ P(\lambda)M - M + T^{-1}A(\lambda)T & P(\lambda) - 2I \end{bmatrix} < 0 \quad (16)$$

where $A(\lambda) = \sum_{i=1}^{q} \lambda_i A_i$, $P(\lambda) = \sum_{i=1}^{q} \lambda_i P_i$, and $\lambda_i \in \Lambda$. The above inequality shows that M and $T^{-1}A(\lambda)T$ are SPR-pair. Based on (13), the above inequality is equivalent to:

$$\begin{bmatrix} (T^{-1}A(\lambda)T)^TP(\lambda)(T^{-1}A(\lambda)T)- P(\lambda) & \star \\ P(\lambda)(T^{-1}A(\lambda)T) + M- T^{-1}A(\lambda)T & P(\lambda) - 2I \end{bmatrix} < 0 \quad (17)$$

Now, multiply the above inequality on the right by $diag(T^{-1}, T^{-1})$ and on the left by $diag(T^{-T}, T^{-T})$:

$$\begin{bmatrix} A(\lambda)^T(T^{-T}P(\lambda)T^{-1})A(\lambda)- T^{-T}P(\lambda)T^{-1} & \star \\ (T^{-T}P(\lambda)T^{-1})A(\lambda)+ T^{-T}MT^{-1} - A(\lambda) & T^{-T}P(\lambda)T^{-1}- 2T^{-T}T^{-1} \end{bmatrix} < 0 \quad (18)$$

Consequently, the closed-loop state matrix of the polytopic system $A(\lambda)$ is stable with a linearly parameter dependent Lyapunov matrix $T^{-T}P(\lambda)T^{-1}$. ■

The convex set of fixed-order stabilizing controller presented in this theorem is an inner convex approximation of the non-convex set of all fixed-order stabilizing controllers for the polytopic system. The quality of this approximation depends on the choice M, the central state matrix, and T, the similarity transformation, which will be considered in next subsection.

C. Choice of the central state matrix and the similarity transformation

In this paper, the non-convexity of the fixed order controller design problem can be overcome by using the instrumental matrices which are the central state matrix M and the similarity transformation T. However, the conservatism of this approach is affected by these matrices and therefore they should be determined in an appropriate way. Generally, one approach to the choice of the instrumental matrices M and T is to use a set of initial fixed-order stabilizing controllers designed for each vertex of the polytopic system and then find the matrix M such that it is SPR-pair with the closed-loop state matrix of each vertex. Now, suppose that $\bar{A}_i$ is the closed-loop state matrix of each vertex with its corresponding controller. Then, a good candidate for the central state matrix M will be a matrix which makes SPR-pair with $T^{-1}\bar{A}_iT$ for $i = 1, \ldots, q$. This matrix and the non-singular matrix T are determined by the equivalent LMIs of (15), which are mentioned in the following remark:

Remark: Based on Lemma 1, (15) is equivalent to the following inequality:

$$\begin{bmatrix} (T^{-1}A_iT)^TP_i(T^{-1}A_iT) - P_i & \star \\ P_i(T^{-1}A_iT) + M - (T^{-1}A_iT) & P_i - 2I \end{bmatrix} < 0 \quad (19)$$

for $i = 1, \ldots, q$.

Then, by multiplying this inequality on the left by $diag(T^{-T}, T^{-T})$ and on the right by $diag(T^{-1}, T^{-1})$, one obtains:

$$\begin{bmatrix} A_i^TP_{T_i}A_i - P_{T_i} & \star \\ P_{T_i}A_i - XA_i + M_T & P_{T_i} - 2X \end{bmatrix} < 0 \quad (20)$$

where

$$\begin{aligned} M_T &= T^{-T}MT^{-1} \\ P_{T_i} &= T^{-T}P_iT^{-1} \\ X &= T^{-T}T^{-1} \end{aligned} \quad (21)$$

for $i = 1, \ldots, q$.

The above inequalities are LMIs with respect to M_T, X, and the positive definite matrices P_{T_i} for $i = 1, \ldots, q$.

The central state matrix M and the similarity transformation T can be chosen as follows:

$$\begin{aligned} M &= T^TM_TT \\ T &= (chol(X))^{-1} \end{aligned} \quad (22)$$

where *chol* is Cholesky factorization and (M_T, X) are a feasible solution to the LMIs in (20) by replacing A_i with $\bar{A}_i$ for $i = 1, \ldots, q$.

If any feasible solution cannot be found for LMIs in (20), different initial controllers can be applied. Another solution is that to decrease the parametric uncertainty domain and determine the matrices M and T. Then, by iterative update on the instrumental matrices, the uncertainty region can be increased.

III. LMI CONDITIONS FOR FIXED-ORDER H_∞ CONTROLLER DESIGN

In this section, the problem of fixed-order H_∞ controller design of the discret-time polytopic system in (1) and (2) is considered. The objective is that the fixed-order controller satisfies an infinity norm bound on the closed-loop transfer function $H_{zw}(\lambda)$.

The idea of SPR-pair matrices can be used as a tool to present a set of LMI conditions based on linearly parameter dependent Lyapunov matrices. This idea will be presented in the following theorem:

Theorem 2: Suppose that a central stable matrix M and a non-singular similarity transformation T are given. Then, the closed-loop system of the discrete-time polytopic system in (1) and (2) with the controller in (5) is stable and $\|H_{zw}(\lambda)\|_\infty^2 < \mu$ if there exist symmetric matrices $P_i > 0$ such that

$$\begin{bmatrix} P_i - M^TP_iM & \star & \star & \star \\ P_iM - M + T^{-1}A_iT & 2I - P_i & \star & \star \\ 0 & (T^{-1}B_i)^T & I & \star \\ CT & 0 & D & \mu I \end{bmatrix} > 0 \quad (23)$$

for $i = 1, 2, \ldots, q$.

Proof: See Appendix I. ■

The above inequalities are LMIs with respect to the controller parameters (A_k, B_k, C_k, D_k), μ, and q symmetric matrices P_i for $i = 1, \ldots, q$.

Lemma 2: The following set of inequalities are equivalent with (23):

$$\begin{bmatrix} P_{Ti} - A_i{}^T P_{Ti} A_i & \star & \star & \star \\ P_{Ti} A_i + M_T - X A_i & 2X - P_{Ti} & \star & \star \\ B_i{}^T M_T - B_i{}^T X A_i & B_i{}^T X & I & \star \\ C & 0 & D & \mu I \end{bmatrix} > 0 \tag{24}$$

for $i = 1, \ldots, q$, where M_T, P_{T_i}, and X are defined in (21).

Proof: Multiply the inequalities in (23) on the left and on the right by the following matrix U_1 and U_1^T, respectively.

$$U_1 = \begin{bmatrix} T^{-T} & T^{-T}M^T - A_i T^{-T} & 0 & 0 \\ 0 & T^{-T} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{25}$$

■

The inequalities in (24) are LMIs in terms of M_T, X, and P_{Ti} for all q vertices.

Now, consider a set of initial fixed-order H_∞ controllers independently designed for each vertex and compute $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, and $\bar{D}_i$ from (7) by replacing the initial controllers for A_k, B_k, C_k and D_k. Then, the central state matrix M and the non-singular matrix T can be obtained through an optimization problem which is minimizing γ subject to the LMIs in (24) (by simply replacing (A_i, B_i, C_i, D_i) with ($\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$)).

The results can be further improved if the resulting controller is used as initial controller to update the central state and the similarity transformation matrices iteratively. This idea will be discussed in the next subsection.

A. An iterative procedure for fixed-order H_∞ controller design

The upper bound μ can be monotonically decreased if the central state and similarity transformation matrices are iteratively updated by applying the previous controller as an initial controller. In this way, at each iteration, two sets of LMIs in (23) and (24) are considered, one in which the instrumental matrices are determined based on the controller of the previous iteration and the other in which the new controller is obtained. The iterative procedure can be summarized by the following steps. To ease the presentation, the inequalities in (23) and (24) are respectively defined as follows:

$$\mathcal{F}_1^i(P_i, K, \mu \mid M, T) < 0 \tag{26}$$

$$\mathcal{F}_2^i(P_{Ti}, M_T, X, \mu \mid K) < 0 \tag{27}$$

for $i = 1, \ldots, q$. The sign $|$ in the arguments of $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$ separates the decision variables and the known parameters in the related LMIs. Therefore, the LMIs in (26) are used to find the controller parameters, $K = (A_c, B_c, C_c, D_c)$ for a given pair of (M, T). In the same way, the LMIs in (27) are used to find M_T, and X for a given controller K.

Step 1: Design an initial controller for each vertex of the polytopic system ($K_i^0; i = 1, \ldots, q$). Put the iteration number $j = 0$ and a large value for $\mu_1^j >> 1$ and a small tolerance for $\epsilon > 0$.

Step 2: Compute M_T^j and X^j from the following optimization problem:

$$\begin{aligned} \mu_2^j = \min \quad & \mu \\ \text{subject to} \quad & \mathcal{F}_2^i(P_{Ti}, M_T^j, X^j, \mu \mid K_i^{j-1}) < 0; \\ & i = 1, \ldots, q \end{aligned} \tag{28}$$

Compute the central state matrix M^j and the similarity transformation matrix T^j using M_T^j, X^j and (22).

Step 3: Solve the following optimization problem to obtain a fixed-order H_∞ controller K:

$$\begin{aligned} \mu_1^j = \min \quad & \mu \\ \text{subject to} \quad & \mathcal{F}_1^i(P_i, K^j, \mu \mid M^j, T^j) < 0; \\ & i = 1, \ldots, q \end{aligned} \tag{29}$$

Step 4: If $\mu_1^{j-1} - \mu_1^j > \epsilon$, use the obtained controller in Step 3 as an initial controller ($K_i^{j+1} = K^{j+1}; i = 1, \ldots, q$) and go to Step 2 with $j = j + 1$, else stop.

It can be proved that this iterative approach leads to monotonic convergence of the upper bound on the H_∞ norm. The proof is based on the fact that (26) and (27) are equivalent inequalities. Therefore, for $j > 1$, K^{j-1} and μ_2^j are always feasible solutions to the optimization problem in Step 3 which guarantee that $\mu_1^j \leq \mu_2^j$. On the other hand, M_T^j, X^j and μ_1^j are always feasible solutions to the optimization problem in Step 2 at iteration $j + 1$. Thus, $\mu_2^{j+1} \leq \mu_1^j$. As a result, $\mu_1^{j+1} \leq \mu_1^j$ which shows that the upper bound μ_1 is not increasing and converges monotonically to a local optimum.

B. Simulation example

Example 1: Consider the following state space model of the system in [12] given by:

$$\begin{aligned} A_g &= \begin{bmatrix} 0 & 0 & -r_1 \\ 1 & 0 & -r_2 \\ 0 & 1 & -r_3 \end{bmatrix}; B_g = \begin{bmatrix} r_4 \\ 1 \\ 0 \end{bmatrix} \\ C_g &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}; \\ C_z &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}; \\ D_{zu} &= 0; \quad D_{zw} = 1; \quad D_w = 1 \end{aligned} \tag{30}$$

with $r_1 = -0.1$, $r_2 = 0.5$, $r_3 = -1.2$, and $r_4 = 0.2$, where all the parameters contain uncertainty up to $\pm 20\%$ of their nominal values, resulting in an unstable polytope with $2^4 = 16$ vertices in a four-dimensional space.

In this example, the problem of low-order H_∞ output feedback controller design is considered. The objective is to design a second-order controller based on Theorem 2 such that it minimizes $\|H_{zw}(\lambda)\|_\infty$.

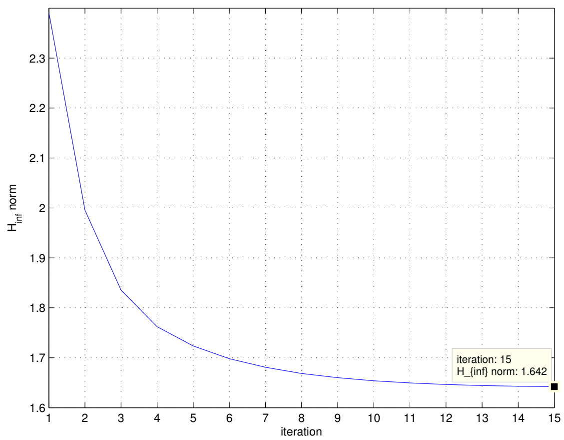

Fig. 1. Evolution of the upper bound of the H_∞ norm versus the iteration number in Example 1

At the first step, initial second-order controllers are designed by using Frequency-Domain Robust Controller Design Toolbox [13] for each vertex of the polytope. Then these controllers are utilized to determine the instrumental matrices M and T using the LMIs in (24). Based on the results of Section III, the iterative algorithm converges to the following controller with $\mu_{min} = 1.6422^2$ after 15 iterations:

$$K(z) = -\frac{0.5413z^2 - 0.2656z + 0.1565}{z^2 + 0.9967z + 0.03198} \quad (31)$$

Figure 1 shows the monotonic decreasing of the upper bound of $\|H_{zw}(\lambda)\|_\infty$ for 15 iterations. Since the state space realization of the system is in the canonical form, the polynomial-based approaches in the literature (e.g. [4], [5], and [14]) can be employed for the comparison purposes. The results of these approaches for two different central polynomials have been mentioned in [14]. Table I summarizes the results.

TABLE I
UPPER BOUND OF $\|H_{zw}(\lambda)\|_\infty$ FOR DIFFERENT APPROACHES IN EXAMPLE 1

Approach	central polynomial 1	central polynomial 2
Results of [4]	2.25	1.95
Results of [5]	2.25	1.95
Results of [14]	1.75	1.75
Results of Theorem 2	1.6422	

IV. LMI CONDITIONS FOR FIXED-ORDER H_2 CONTROLLER DESIGN

The main objective of this section is to propose a set of LMIs for fixed-order stabilizing controller design of the polytopic systems in (1) and (2) which satisfies H_2 performance $\|H_{zw}(\lambda)\|_2^2 < \upsilon$.

It is assumed that either matrix D_{zu} or matrix D_w in (1) are equal to zero. In what follows $D_w = 0$ is considered. Therefore, the controller matrices appear only in matrices $A(\lambda)$ and C.

The next theorem shows that how the idea of SPR-pair matrices can be used to present a convex set of fixed-order H_2 controllers.

Theorem 3: Suppose that a central stable matrix M and a nonsingular similarity transformation T are given. Then, the closed-loop system of the polytopic system in (1) and (2) with the controller in (5) is stable and $\|H_{zw}(\lambda)\|_2^2 < \upsilon$ if there exist symmetric matrices $P_i > 0$ and $W_i > 0$ such that:

$$\begin{bmatrix} P_i - M^T P_i M & \star & \star \\ P_i M - M + T^{-1} A_i T & 2I - P_i & \star \\ C_i T & 0 & I \end{bmatrix} > 0$$
$$\begin{bmatrix} W_i & \star & \star \\ P_i T^{-1} B & P_i & \star \\ D_i & 0 & I \end{bmatrix} > 0 \quad (32)$$
$$trace(W_i) < \upsilon$$

for $i = 1, 2, \ldots, q$.

Proof: See Appendix II. ∎

The inequalities in (32) are LMIs with respect to the controller parameters (A_k, B_k, C_k, D_k), υ, and q symmetric matrices (P_i, W_i) for $i = 1, \ldots, q$.

The quality of this approach is dependent of the choice of the state matrix M and the non-singular matrix T which can be acquired based upon the following set of LMIs:

$$\begin{bmatrix} P_{Ti} - A_i{}^T P_{Ti} A_i & \star & \star \\ P_{Ti} A_i - X A_i + M_T & 2X - P_{Ti} & \star \\ C_i & 0 & I \end{bmatrix} > 0$$
$$\begin{bmatrix} W_i & \star & \star \\ P_{Ti} B_i & P_{Ti} & \star \\ D_i & 0 & I \end{bmatrix} > 0 \quad (33)$$
$$trace(W_i) < \upsilon$$

for $i = 1, \ldots, q$.

Lemma 3: The inequalities in (32) and (33) are equivalent.

Proof: Pre and post multiply the inequalities in (32) with the following matrix U_2 and U_2^T, respectively:

$$U_2 = \begin{bmatrix} T^{-T} & T^{-T} M^T - A_i^T T^{-T} & 0 \\ 0 & T^{-T} & 0 \\ 0 & 0 & I \end{bmatrix} \quad (34)$$

∎

Remarks:

1) The matrices M and T can be obtained by (22) where (M_T, X) is a feasible solution of an optimization problem which is minimizing υ subject to the LMI conditions in (33) (by simply replacing (A_i, B_i, C_i, D_i) with ($\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$)).
2) It can be shown that the upper bound υ is monotonically decreased by an iterative algorithm for update on the instrumental matrices M and T.
3) When $D_w \neq 0$ and $D_{zu} = 0$, the unknown controller matrices appear in A and B and then the second set of inequalities in (32) is not an LMI with respect to

unknown variables. In this case, based on the fact that $\|H_{zw}(\lambda)\|_2 = \|H_{zw}^T(\lambda)\|_2$, the dual problem which is the minimization of two norm of $H_{zw}^T(\lambda)$ with the realization $(A^T(\lambda), C^T, B^T(\lambda), D^T)$ is considered.

4) For the case when $D_w \neq 0$ and $D_{zu} \neq 0$, the second inequality in (32) can be formulated as a set of LMIs by using an instrumental matrix Y as follows:

$$\begin{bmatrix} W_i & \star & \star \\ Y^T T^{-1} B_i & Y + Y^T - P_i & \star \\ D & 0 & I \end{bmatrix} > 0 \qquad (35)$$

for $i = 1, \ldots, q$. It can be easily proved by applying Schur Complement Lemma [15] to (35) and considering the fact that $(Y - P_i)^T P_i^{-1} (Y - P_i) \geq 0$.

5) Mixed H_∞ and H_2 control design can be addressed by gathering the LMIs in (23) and (32) in a single set of LMIs without using a common Lyapunov matrix and unique instrumental matrices (central state and similarity transform matrices) for both objectives.

A. Simulation example

Example 2: Consider the following forth-order unstable polytopic system in [16]:

$$A_g = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & p_1 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & p_2 \end{bmatrix}$$
$$B_g = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & p_3 \end{bmatrix}; C_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}; C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; D_{zw} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; D_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (36)$$

with $p_1 = 0.7901$, $p_2 = 0.8604$, and $p_3 = 0.0936$.

Case a: No uncertainty in the system

First, by using the zero-order hold (*ZOH*) transformation, the *hinfstruct* command in MATLAB (by setting the target gain to 4) has been applied to design an initial static output feedback controller for the system. Then, by applying the method in Section IV and after 25 iteration, the following static output feedback is designed such that $\|H_{zw}\|_2 < 0.2727$:

$$K_1 = \begin{bmatrix} -0.3506 & -0.4405 \\ -0.3039 & -0.0007 \end{bmatrix} \qquad (37)$$

The convergence of the algorithm is illustrated in figure 2. The results are compared with the results in [16], (example 2), where $\|H_{zw}\|_2 < 0.4243$.

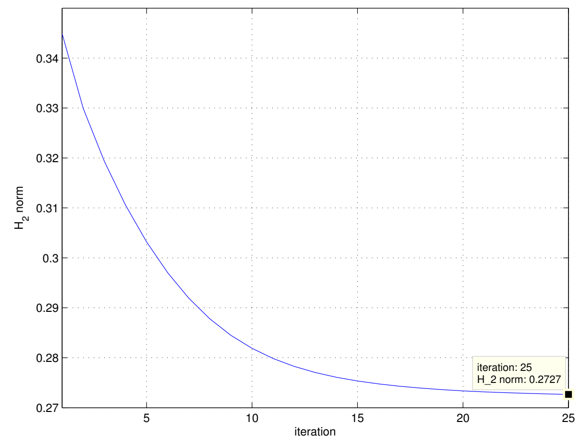

Fig. 2. Convergence of the upper bound of H_2 norm in Example 2 (case a)

Case b: Polytope with 2^3 vertices

In this case, we assume that there are uncertainty in the parameters p_1, p_2 and p_3 up to $\pm 8\%$ of their nominal values. The *hinfstruct* command in MATLAB (by setting the target gain to 4) has been used to design initial controllers for each vertex of the polytope in (36). These controllers are utilized to find a feasible solution M and T of (33). Finally, the results of Theorem 3 provides the following controller:

$$K_2 = \begin{bmatrix} -0.9915 & -0.6291 \\ -0.1881 & -0.02591 \end{bmatrix} \qquad (38)$$

The results can be improved more by an iterative approach in which M and T are updated based on the controller in the last iteration as an initial controller. Figure 3 shows the upper bound of $\|H_{zw}\|_2$ for 5 iterations. After 5 iterations, the upper bound $\upsilon_{min} = 0.4187^2$ is obtained for the polytopic system with the following controller:

$$K_3 = \begin{bmatrix} -0.9243 & -0.7063 \\ -0.3921 & -0.0254 \end{bmatrix} \qquad (39)$$

In this example, the proposed approach provides the results which are even better than the results in [16] where a static output feedback has been designed for the system in (36) without uncertainty.

V. Conclusion

In this paper, LMI conditions for fixed-order H_∞ and H_2 output feedback control design of discrete-time systems with polytopic uncertainty has been proposed. The conditions are based on the new concept of SPR-pair matrices which are utilized as a tool to convexify the stability conditions as well as the H_∞ and H_2 performance specifications. It has been shown that the proposed method monotonically converges to a suboptimal solution by iterative update on the instrumental matrices. The simulation examples have demonstrated the effectiveness of the proposed method.

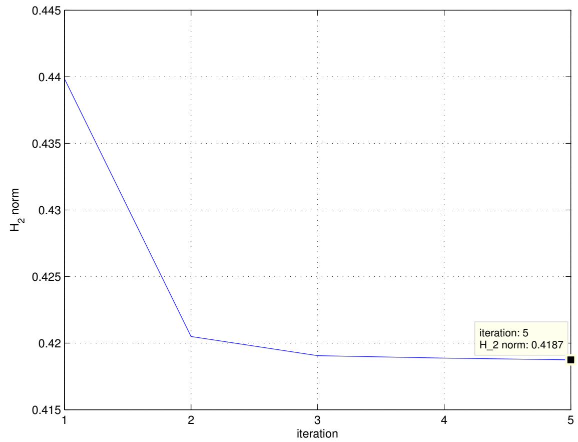

Fig. 3. Evolution of the upper bound of the two norm versus the iteration number in Example 2 (case b)

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APPENDIX I
PROOF OF THEOREM 2

Proof: By convex combination of (23) for all vertices, the following inequality can be obtained:

$$\begin{bmatrix} P(\lambda)-M^TP(\lambda)M & \star & \star & \star \\ P(\lambda)M-M+T^{-1}A(\lambda)T & 2I-P(\lambda) & \star & \star \\ 0 & (T^{-1}B(\lambda))^T & I & \star \\ CT & 0 & D & \mu I \end{bmatrix} > 0 \tag{40}$$

where $A(\lambda)=\sum_{i=1}^q \lambda_i A_i$, $B(\lambda)=\sum_{i=1}^q \lambda_i B_i$, $P(\lambda)=\sum_{i=1}^q \lambda_i P_i$, and $\lambda \in \Lambda$. Then, by pre and post multiplication of the above matrix with the following matrix U_3 and U_3^T, respectively, Bounded Real Lemma can be easily reached which means $\|H_{zw}(\lambda)\|_\infty^2 < \mu$.

$$U_3 = \begin{bmatrix} T^{-1} & T^{-T}M^TT^TT^{-1} & 0 & -\mu^{-1}C^T \\ 0 & B^{cl}(\lambda)^T T^{-1} & -I & \mu^{-1}D^T \end{bmatrix} \tag{41}$$

■

APPENDIX II
PROOF OF THEOREM 3

Proof: Convex combination of (32) for all vertices leads to the following inequalities:

$$\begin{bmatrix} P(\lambda)-M^TP(\lambda)M & \star & \star \\ P(\lambda)M-M+T^{-1}A(\lambda)T & 2I-P(\lambda) & \star \\ CT & 0 & I \end{bmatrix} > 0 \tag{42}$$

$$\begin{bmatrix} W(\lambda) & \star & \star \\ P(\lambda)T^{-1}B & P(\lambda) & \star \\ D & 0 & I \end{bmatrix} > 0 \tag{43}$$

$$trace(W(\lambda)) < \upsilon \tag{44}$$

where $A(\lambda)=\sum_{i=1}^q \lambda_i A_i$, $W(\lambda)=\sum_{i=1}^q \lambda_i W_i$, $P(\lambda)=\sum_{i=1}^q \lambda_i P_i$, and $\lambda \in \Lambda$.

By pre-multiplication of (42), (43), and (44) with the following matrices U_4, U_5, and U_6, respectively, and then post-multiplication with their transposes

$$U_4 = \begin{bmatrix} T^{-T} & T^{-1}M^T - A^TT^{-1} \end{bmatrix} \tag{45}$$

$$U_5 = diag(I, T^{-1}, I) \tag{46}$$

$$U_6 = 1 \tag{47}$$

the following inequalities are obtained:

$$\begin{gathered} \begin{bmatrix} P(\lambda) & \star & \star \\ P(\lambda)A(\lambda) & P(\lambda) & \star \\ C & 0 & I \end{bmatrix} > 0 \\ \begin{bmatrix} W(\lambda) & \star & \star \\ P(\lambda)B & P(\lambda) & \star \\ D & 0 & I \end{bmatrix} > 0 \\ trace(W(\lambda)) < \upsilon \end{gathered} \tag{48}$$

Therefore, the square of the two-norm of H_{zw} with the realization $(T^{-1}A(\lambda)T, T^{-1}B, CT, D)$ is less than υ. ■