A global branch of solutions to a semilinear equation on an unbounded interval

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(MS received 4 April 1985)

Synopsis
For a semilinear second order differential equation on \((0, \infty)\), conditions are given for the bifurcation and asymptotic bifurcation in \(L^p\) of solutions to the Neumann problem. Bifurcation occurs at the lowest point of the spectrum of the linearised problem. Under stronger hypotheses, there is a global branch of solutions. These results imply similar conclusions for the same equation on \(\mathbb{R}\) with appropriate symmetry.

1. Introduction
We consider the following Neumann problem:
\[
\begin{align*}
    u''(x) + \lambda u(x) + q(x)f(u(x), u'(x)) &= 0 \quad \text{for } x > 0, \\
    u'(0) = \lim_{x \to \infty} u(x) &= 0,
\end{align*}
\]
where the functions \(q\) and \(f\) satisfy:

(H1) \(q \in C(\mathbb{R}_+, \mathbb{R})\) and \(\lim_{x \to \infty} q(x) = L\) with \(L > 0\);

(H2) \(f \in C^1(\mathbb{R}^2, \mathbb{R})\) with \(f(0, 0) = 0\) and \(\text{grad} f(0, 0) = (0, 0)\);

(H3) there exist positive constants \(a\) and \(A\) such that
\[
    k^{-2}D_1f(k^a s, k^{a+1} t) \to A |s|^{2/a},
\]
\[
    k^{-1}D_2f(k^a s, k^{a+1} t) \to 0 \quad \text{as } k \to 0^+,
\]
uniformly for \((s, t)\) in bounded subsets of \(\mathbb{R}^2\).

A classical solution to (N) is a pair \((\lambda, u)\) where \(\lambda \in \mathbb{R}, u \in C^2(\mathbb{R}_+, \mathbb{R})\) and (N) is satisfied. It is convenient to reformulate this problem using Sobolev spaces [1, Chap. VIII]. Let
\[
    X = \{u \in W^{2,1}((0, \infty)): u'(0) = 0\}
\]
with the norm
\[
    \|u\|_X = |u|_1 + |u'|_1 + |u''|_1, \quad \text{for } u \in X.
\]
where \(|u|_p\) denotes the usual norm in \(L^p((0, \infty)) = L^p\) for \(1 \leq p \leq \infty\). Then
\[
    X \subset C^1(\mathbb{R}_+, \mathbb{R}) \cap L^p \quad \text{for } 1 \leq p \leq \infty \quad \text{and} \quad \lim_{x \to \infty} u(x) = \lim_{x \to \infty} u'(x) = 0 \quad \text{for all } u \in X.
\]
Hence we see that \((\lambda, u)\) is a classical solution to (N) provided that \((\lambda, u) \in \mathbb{R} \times X\) and (N) is satisfied almost everywhere on \(\mathbb{R}_+\).
This paper is concerned with the bifurcation of solutions to (N) from the point 
(0, 0) in \( R \times L^p \). The question has already been studied in several contexts when \( f \) 
is independent of \( u' \). Variational methods are used in [6,7], the topological 
degree is used in [9,10] and in [12] the case where \( q(x) = 1 \) for all \( x \geq 0 \) is solved 
by quadrature. From these contributions we know that when 

(i) \( q \) is non-increasing and satisfies (H1), 
(ii) \( f(s, t) = |s|^{2/a} s \) for all \( (s, t) \in R^2 \),

there is bifurcation in \( L^p \) if and only if \( ap > 1 \).

We show how a suitable scaling of the variables can be used to reduce (N) to a 
situation where the implicit function theorem establishes the bifurcation of a 
continuous branch of solutions to (N) in \( R \times L^p \) provided that \( ap > 1 \). Scaling has 
been used in a similar way to deal with bifurcation for certain integral equations 
involving convolutions [3,4]. Recently scaling has also been used to establish 
bifurcation at an eigenvalue where the linearisation is not a Fredholm operator 
[5]. We recall, however, that the linearisation of (N) has no eigenvalues in \( L^p \) for 
\( p \geq 1 \).

Under the more restrictive assumptions (L1) to (L3), we are able to prove that 
this branch of solutions can be extended to a curve parametrised by \( \lambda \) for all 
\( \lambda \in (-\infty, 0) \). Furthermore, the functions \( u \) corresponding to solutions on this curve 
are positive.

To state our results for (N), we first introduce an “asymptotic limit for (N)”: 

\[
\begin{align*}
    v''(x) - v(x) + B |v(x)|^{2/a} v(x) &= 0 \quad \text{for} \quad x > 0, \\
    v'(0) &= 0 \quad \text{and} \quad \lim_{x \to \infty} v(x) = 0,
\end{align*}
\]

\[\text{(N)} \]

where \( B = aAL/(2 + a) \) and \( L, a \) and \( A \) are the constants appearing in (H1) and 
(H3). An elementary phase-plane analysis shows that (N)\(_\infty\) has a unique solution, 
denoted by \( v_0 \), and that \( v_0 \) is positive and decreasing.

**THEOREM 1. (Bifurcation).** Let the conditions (H1), (H2) and (H3) be satisfied. 
Then there exist \( \eta > 0 \) and \( v \in C([0, \eta], X) \) such that \( v(0) = v_0 \) and, for \( 0 < k < \eta \), 
\((-k^2, u_k)\) is a non-trivial classical solution to (N) where \( u_k(x) = k^{a} v(k)(kx) \) for 
\( x \geq 0 \).

**Remark.** For \( 1 \leq p \leq \infty \), \( |u_k|_p = k^{a-(1/p)} |v(k)|_p \) and \( |v(k)|_p \to |v_0|_p \) as \( k \to 0^+ \). 
Furthermore, \( k \mapsto (-k^2, u_k) \) is a continuous curve in \( R \times L^p \) for \( 1 \leq p \leq \infty \) and 
\((-k^2, u_k) \to (0, 0) \) as \( k \to 0^+ \) provided that \( ap > 1 \).

Under stronger assumptions, we can improve this local result and show that the 
above curve extends globally.

(L1) \( q \in C^1(R_+, R) \) with \( q'(x) \leq 0 \) for all \( x \geq 0 \) and \( \lim_{x \to \infty} q(x) = L \) where \( L > 0 \).

(L2) \( h \in C^1(R_+, R) \) with \( h(0) = h'(0) = 0 \) and \( s^2h'(s) > sh(s) > rH(s) > 0 \) for all 
\( s > 0 \) where \( r > 2 \) and \( H(s) = \int_0^s h(t) \, dt \).

(L3) There exist positive constants \( a \) and \( A \) such that \( k^{-2}h'(k^{a}s) \to As^{2/a} \) as 
\( k \to 0^+ \).

Apart from permitting a global analysis, these hypotheses also ensure that the
solutions on the branch have the same qualitative behaviour as $v_0$. Let

$$K = \{ u \in X : u(x) > 0 \text{ and } u'(x) < 0 \text{ for all } x > 0 \}.$$ 

**Theorem 2.** (Global continuation). Let the conditions (L1), (L2) and (L3) be satisfied and set $f(s, t) = h(|s|)$ for all $(s, t) \in \mathbb{R}^2$. Then there exists $u \in \mathcal{C}^1((-\infty, 0), X)$ such that for all $\lambda < 0, (\lambda, u(\lambda))$ is a (non-trivial) solution to (N) and $u(\lambda) \in K$. Furthermore, for $0 < \sqrt{-\lambda} < \eta$, $u(\lambda) = u_{\sqrt{-\lambda}}$ where $u_k$ is the solution given in Theorem 1.

The local result is proved in Section 2 and the global continuation is established in Section 3.

Remarks. 1. Solutions to the Neumann problem (N) can be used to construct solutions to the following related problem:

$$
\begin{align*}
\left. \begin{array}{l}
u''(x) + \lambda \nu(x) + q(x)\nu(x), \nu'(x) = 0 \quad \text{for } x \in \mathbb{R}, \\
\lim_{x \to -\infty} \nu(x) = \lim_{x \to \infty} \nu(x) = 0,
\end{array}\right\} \\
\text{(D)}
\end{align*}
$$

provided that

(a) $q \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is even and satisfies (H1),
(b) $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies (H2) and (H3) with $f(s, t) = f(s, -t)$ for all $(s, t) \in \mathbb{R}^2$.

In fact, under these conditions a solution $(\lambda, \nu)$ to (N) is made into a solution to (D) by simply extending $\nu$ to be an even function on $\mathbb{R}$. Thus results similar to Theorems 1 and 2 hold for the problem (D) under the assumptions (a) and (b). When $q$ is not even, the problem (D) cannot be reduced to (N) and the situation is much more complicated [8].

2. The method of scaling can also be applied to the $N$-dimensional generalisation of (D):

$$
\Delta \nu(x) + \lambda \nu(x) + q(x)\nu(x) = 0 \quad \text{for } x \in \mathbb{R}^N,
$$

provided that $q$ is radially symmetric. This remark will be amplified elsewhere.

### 2. Bifurcation by scaling

In this section we prove Theorem 1 by reducing the problem (N) to a situation in which the implicit function can be applied.

**Lemma 2.1.** Let the function $f$ satisfy the conditions (H2) and (H3). Given $\varepsilon > 0$ and a bounded subset $D$ of $\mathbb{R}^2$, there exists $\delta > 0$ such that

$$
|k^{-(2+a)}f(k^a, x, k^{a+1}t) - \frac{aA}{2 + a} | |s|^{2/a} |t| < \varepsilon \{ |s| + |t| \}
$$

for $0 < k < \delta$ and $(s, t) \in D$. 

Proof. For \( k > 0 \) and \((s, t) \in \mathbb{R}^2\),
\[
\left| k^{-(2+a)} f(k^a s, k^{a+1} t) - \frac{aA |s|^{2/a}}{2+a} \right|
= \left| \int_0^1 \left\{ k^{-(2+a)} \frac{d}{dr} f(rk^a s, rk^{a+1} t) - A |rs|^{2/a} s \right\} dr \right|
\leq \int_0^1 |k^{-2} D_1 f(rk^a s, rk^{a+1} t) - A |rs|^{2/a} f(rk^a s, rk^{a+1} t) | dr |s|
+ \int_0^1 |k^{-1} D_2 f(rk^a s, rk^{a+1} t) | dr |t|.
\]

The result now follows from (H3).
For \((k, u) \in \mathbb{R} \times X\), we define a function \( F \) as follows:
\[
F(k, u)(x) = \begin{cases} 
q(x/|k|) |k|^{-(2+a)} f(|k|^a u(x), |k|^{a+1} u'(x)) & \text{if } k \neq 0, \\
B |u(x)|^{2/a} u(x) & \text{if } k = 0,
\end{cases}
\]
where \( B = aAL/(2+a) \).

**Lemma 2.2.** Let the conditions (H1), (H2) and (H3) be satisfied.
(a) \( F \) maps \( \mathbb{R} \times X \) continuously into \( L^1 \).
(b) For each \( k \in \mathbb{R} \), \( F(k, \cdot) : X \to L^1 \) is Fréchet differentiable and for \( u, v \in X \),
\[
D_u F(k, u)v(x) = \begin{cases} 
q(x/|k|) |k|^{-2} D_1 f(|k|^a u(x), |k|^{a+1} u'(x)) v(x) & \text{if } k \neq 0, \\
+ |k|^{-1} D_2 f(|k|^a u(x), |k|^{a+1} u'(x)) v'(x) & \text{if } k = 0.
\end{cases}
\]
(c) \( D_u F \) maps \( \mathbb{R} \times X \) continuously into the Banach space of all bounded linear operators from \( X \) into \( L^1 \).

**Proof.** We recall that \( X \) is continuously embedded in \( W^{1,\infty}(0, \infty) \). The lemma is then established in a standard manner using (H3) and Lemma 2.1.

**Lemma 2.3.** Let the conditions (H1), (H2) and (H3) be satisfied. Let \( u(x) = k^n v(kx) \) for \( x \geq 0 \) and \( k > 0 \). The following statements are equivalent.
(i) \((-k^2, u) \in \mathbb{R} \times X\) is a solution to (N).
(ii) \( v \in X \) and \( v''(x) - v(x) + F(k, v)(x) = 0 \) for \( x > 0 \).

**Proof.** Trivial.

**Proof of Theorem 1.** Let \( Tu = u'' - u \) and let \( G(k, u) = Tu + F(k, u) \). Then \( T : X \to L^1 \) is an isomorphism and \( G : \mathbb{R} \times X \to L^1 \) is continuous. Furthermore, for \( u, v \in X \) and \( k \in \mathbb{R} \), \( D_u G(k, u)v = Tv + D_u F(k, u)v \). Thus \( G(0, v_0) = 0 \) where \( v_0 \) is the unique solution to (N)\(\infty \) in \( X \).

Hence, by Lemma 2.2 and the implicit function theorem (see for example [2, p. 222]), it is sufficient to show that \( D_u G(0, v_0) : X \to L^1 \) is an isomorphism.

Let \( C v(x) = Al v_0(x)^{2/a} v(x) \) for \( v \in X \). Then \( C : X \to L^1 \) is a compact linear operator and \( D_u G(0, v_0) = T + C \). Since \( T : X \to L^1 \) is an isomorphism, we need only show that \( T + C \) is injective. For this, we suppose that \( v \in X \) and \((T + C)v = 0\).
Solutions to a semilinear equation

Then

$$v''(x) - v(x) + AL |v_0(x)|^{2/a} v(x) = 0 \quad \text{for } x > 0. \quad (2.1)$$

From (2.1) and (N)_\infty, we have that \( \int_0^\infty |v_0(x)|^{2/a} v_0(x) v(x) \, dx = 0 \) and so \( v \) has at least one zero, denoted by \( z \), in \( (0, \infty) \). On setting \( w(x) = v_0(x) \), we have that

$$w''(x) - w(x) + AL |v_0(x)|^{2/a} w(x) = 0 \quad \text{for } x > 0 \quad (2.2)$$

since \( v_0 \) satisfies \( (N)_\infty \). From (2.1), we obtain

$$-v'(z)w(z) + \int_z^\infty \{ -v'(x)w'(x) - v(x)w(x) + AL |v_0(x)|^{2/a} v(x)w(x) \} \, dx = 0$$

and from (2.2),

$$\int_z^\infty \{ -v'(x)w'(x) - v(x)w(x) + AL |v_0|^{2/a} v(x)w(x) \} \, dx = 0.$$

Hence, \( v'(z) = 0 \) and so, by (2.1), we must have \( v(x) = 0 \) for all \( x \geq 0 \).

This completes the proof of the theorem.

3. Global continuation

Throughout this section we suppose that \( f: \mathbb{R}^2 \to \mathbb{R} \) is defined by

$$f(s, t) = \begin{cases} h(|s|) \frac{|s|}{s} & \text{for } (s, t) \in \mathbb{R}^2 \quad \text{with } s \neq 0, \\ 0 & \text{for } s = 0 \quad \text{and } t \in \mathbb{R}, \end{cases}$$

where the function \( h \) satisfies the conditions (L2) and (L3). It is easily seen that \( f \) satisfies the hypotheses (H2) and (H3) and the problem (N) can be written as

$$\begin{cases} u''(x) + \lambda u(x) + q(x) g(|u(x)|) u(x) = 0 \quad \text{for } x > 0, \\ u'(0) = 0 \quad \text{and } \lim_{x \to \infty} u(x) = 0, \end{cases}$$

where \( g(s) = s^{-1} h(s) \) for \( s > 0 \). For the proofs which follow, we note some simple consequences of the assumptions (L2) and (L3).

(i) The function \( s^{-\gamma} H(s) \) is increasing for \( s \) in \( \mathbb{R}_+ \).

(ii) There is an increasing function \( g \in C(\mathbb{R}_+, \mathbb{R}) \) such that \( sg(s) = h(s) \) for all \( s \geq 0 \), \( g(0) = 0 \) and \( \lim_{s \to \infty} g(s) = \infty \).

(iii) On setting \( \theta = \gamma(r - 2) \), we have that

$$0 < g(s) < \theta \{ g(s) - 2s^{-2} H(s) \} \quad \text{for all } s > 0.$$ 

(iv) Setting \( j(s) = s^{-\gamma} H(s) \) for \( s > 0 \), we have that \( j \) is increasing on \( (0, \infty) \) with

$$\lim_{s \to 0} j(s) = 0 \quad \text{and} \quad \lim_{s \to \infty} j(s) = \infty.$$ 

(v) \( f(s, t) = g(|s|) s \) for all \( (s, t) \in \mathbb{R}^2 \).

We begin by showing that the branch of solutions of (N) given by Theorem 1 lies in the set \( K \). Then we show that a branch of solutions cannot leave \( K \). Finally,
by a priori estimates and the implicit function theorem, we prove that the branch
can be extended globally to cover \((-\infty, 0)\).

**Lemma 3.1.** Let the conditions (L1), (L2) and (L3) be satisfied and let \(v \in C([0, \eta], X)\) be the function given by Theorem 1. There exists \(k_0 > 0\) such that \(v(k) \in K\) for \(0 < k < k_0\).

**Proof.** From the phase-plane for \((N)_\infty\), we see that \(v(0) = v_0 \in K\). For \(0 < k < \eta\), \(v(k) \in X\) and satisfies

\[v''(x) + \left\{-1 + q(x/k)k^{-2}g(k^a|v(x)|)\right\}v(x) = 0 \quad \text{for} \quad x > 0. \tag{3.1}\]

By adapting Lemma 2.1 to the stronger hypotheses we obtain the following. Given \(\epsilon > 0\) and a bounded subset \(D\) of \(R\), there exists \(\delta > 0\) such that

\[
\left| k^{-2}g(k^a|s|) - \frac{aA}{2 + a} |s|^{2/a} \right| < \epsilon
\]

for \(0 < k < \delta\) and \(s \in D\).

Thus, since \(\lim_{x \to \infty} v_0(x) = 0\), there exists \(z > 0\) such that

\[
\left\{-1 + \frac{q(0)aA |v_0(x)|^{2/a}}{2 + a}\right\} < -\frac{1}{2} \quad \text{for all} \quad x \geq z.
\]

Hence there is an open neighbourhood \(U\) of \((0, v_0)\) in \(R \times X\) such that

\[
\left\{-1 + q(x/k)k^{-2}g(k^a|v(x)|)\right\} < -\frac{1}{4} \quad \text{for all} \quad x \geq z.
\]

provided that \((k, v) \in U\).

If \((k, v) \in U\) and (3.1) is satisfied, it follows that

\[
v'(x)v(x) + \int_x^\infty v'(y)^2 \, dy = \int_x^\infty \left\{-1 + q(y/k)k^{-2}g(k^a|v(y)|)\right\}v(y)^2 \, dy
\]

\[< -\frac{1}{4} \int_x^\infty v(y)^2 \, dy \quad \text{for all} \quad x \geq z.
\]

This proves that \(v'(x)v(x) < 0\) for all \(x \geq z\), and since \(v_0 \in K\), we can conclude that \(v(x) > 0\) and \(v'(x) < 0\) for all \(x \geq z\), provided that \((k, v) \in U\) and satisfies (3.1). On choosing a sufficiently small neighbourhood \(U\) of \((0, v_0)\), the result now follows from the continuous embedding of \(X\) in \(C^1(R_+, R)\).

**Lemma 3.2.** Let the conditions (L1), (L2) and (L3) be satisfied.

(a) The solutions to \((N)\) form a closed subset of \(R \times X\).

(b) If \((\lambda, u)\) is a solution to \((N)\) with \(\lambda < 0\) and \(u \in \bar{K}\) (the closure of \(K\) in \(X\)), then \(u \in K \cup \{0\}\).

(c) If \((\lambda, u)\) is a solution to \((N)\) with \(\lambda < 0\) and \(u \in K\), there is an open

neighbourhood \(U\) of \((\lambda, u)\) in \(R \times X\) such that \(v \in K\) whenever \((\mu, v) \in U\) and satisfies \((N)\).

**Proof.** (a) Trivial.

(b) Suppose that \(u \neq 0\). Since \(u\) satisfies \((N)\) it can only have simple zeros. This
implies that \( u(x) > 0 \) for all \( x \geq 0 \). On setting \( w(x) = u'(x)/u(x) \), we find that

\[
w'(x) = \frac{u''(x)}{u(x)} - \left( \frac{u'(x)}{u(x)} \right)^2
\]

and

\[
u'(x)^2 + \lambda u(x)^2 + 2q(x)H(u(x)) = -\int_x^\infty 2q'(y)H(u(y)) \, dy \quad \text{for} \quad x \geq 0.
\]

Thus,

\[
w'(x) = q(x)\left\{ \frac{2H(u(x))}{u(x)^2} - g(u(x)) \right\} + \frac{1}{u(x)^2} \int_x^\infty 2q'(y)H(u(y)) \, dy
\]

\[
< -\frac{1}{\theta} q(x)g(u(x)) \quad \text{(by (iii))}
\]

\[
< 0.
\]

Since \( w(0) = 0 \), it follows that \( u'(x) < 0 \) for all \( x > 0 \) and so \( u \in K \).

(c) For \( u \in K \), we have \( u(x) > 0 \) for \( x \geq 0 \) and since \( u \) satisfies (N) we also have that \( u''(x) + (\lambda + q(x)g(u(x)))u(x) = 0 \) for \( x > 0 \). Furthermore, \( \lim_{x \to \infty} u(x) = 0 \) and so there exists \( z > 0 \) such that \( \lambda + q(x)g(u(x)) \leq \frac{1}{2} \lambda < 0 \) for all \( x \geq z \). The result is now established in the same way as Lemma 3.1.

**Lemma 3.3.** Let the conditions (L1), (L2) and (L3) be satisfied. Let \( (\lambda, u) \) be a solution to (N) with \( \lambda < 0 \) and \( u \in K \). Set \( k = \sqrt{-\lambda} \).

(a) \( 0 < 2LH(u(x)) \leq k^2 u(x)^2 - u'(x)^2 \leq 2q(x)H(u(x)) \) for \( x \geq 0 \).

(b) \( \lim_{x \to \infty} u'(x)/u(x) = -k \) and, for all \( \varepsilon > 0 \), \( \lim_{x \to \infty} e^{(k-\varepsilon)x}u(x) = 0 \).

**Proof.** (a) By (N), \( u'(x)^2 - k^2 u(x)^2 = \int_x^\infty 2q(y)H(u(y)) \, dy \) for \( x \geq 0 \). But \( q'(y) \leq 0 \) and \( H(u(y))' \leq 0 \) by (L1), (L2) and the assumption that \( u \in K \). Hence we obtain,

\[
-2q(x)H(u(x)) \leq \int_x^\infty 2q(y)H(u(y)) \, dy \leq -2LH(u(x)).
\]

This proves (a).

(b) By (L1) and (L2), \( \lim_{x \to \infty} q(x) = L \) and \( \lim_{x \to \infty} j(x) = 0 \). From (a) it now follows that \( \lim_{x \to \infty} u'(x)/u(x) = -k \) and, given \( \varepsilon > 0 \), there exists \( z \geq 0 \) such that \( u'(x) \leq (k + \varepsilon)u(x) \) for all \( x \geq z \). This implies that \( e^{(k-\varepsilon)x}u(x) \) is a decreasing function of \( x \) on \([z, \infty)\). The proof is complete.

**Lemma 3.4.** Let the conditions (L1), (L2) and (L3) be satisfied. There exist increasing functions \( A \) and \( B \in C((-\infty, 0), R) \) such that

\[
A(\lambda) \leq B(\lambda) \quad \text{for all} \quad \lambda < 0,
\]

\[
0 < A(\lambda) \leq |u|_* = u(0) \leq B(\lambda)
\]

and

\[
A(\lambda) \leq \|u\|_* \leq B(\lambda) \left\{ \frac{1}{\sqrt{-\lambda}} + \theta2\sqrt{-\lambda} \right\}
\]
whenever \((\lambda, u)\) is a solution to \((N)\) with \(\lambda < 0\) and \(u \in K\). Furthermore, \[
\lim_{\lambda \to -\infty} A(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to 0} B(\lambda) = 0.
\]

**Proof.** Let \(A(\lambda) = j^{-1}(-\lambda/2q(0))\) and \(B(\lambda) = j^{-1}(-\lambda/2L)\). From Lemma 3.3(a), we see that \(2Lj(u(0)) \leq -\lambda \leq 2q(0)j(u(0))\) and hence \(A(\lambda) \leq u(0) \leq B(\lambda)\) whenever \((\lambda, u)\) is a solution to \((N)\) with \(\lambda < 0\) and \(u \in K\).

Now setting \(w(x) = u'(x)/u(x)\) as in Lemma 3.2(b), we obtain \(w'(x) < -(1/\theta)q(x)g(u(x))\) for all \(x > 0\), with \(w(0) = 0\) and \(\lim_{x \to \infty} w(x) = -\sqrt{-\lambda}\), by Lemma 3.3(b). Hence, \[
\int_0^\infty q(x)g(u(x))\,dx \leq \theta \sqrt{-\lambda}
\]
and, by \((N)\),
\[
-\lambda \int_0^\infty u(x)\,dx = \int_0^\infty q(x)g(u(x))u(x)\,dx \leq u(0)\theta \sqrt{-\lambda} \leq B(\lambda)\theta \sqrt{-\lambda}.
\]
Thus \(|u|_1 \leq B(\lambda)\theta \sqrt{-\lambda}\) and
\[
|u''|_1 \leq -\lambda |u|_1 + \int_0^\infty q(x)g(u(x))u(x)\,dx \leq -2\lambda |u|_1 \leq 2B(\lambda)\theta \sqrt{-\lambda}.
\]

On the other hand,
\[
A(\lambda) \leq u(0) = -\int_0^\infty u'(x)\,dx = |u'|_1 \leq B(\lambda).
\]

Thus, we have that
\[
||u||_x = |u|_1 + |u'|_1 + |u''|_1 \leq |u'|_1 \leq A(\lambda)
\]
and
\[
||u||_x \leq \frac{B(\lambda)\theta}{\sqrt{-\lambda}} + B(\lambda) + 2B(\lambda)\theta \sqrt{-\lambda}
\]
\[
= B(\lambda)\theta \left\{ \frac{1}{\sqrt{-\lambda}} + \frac{1}{\theta} + 2\sqrt{-\lambda} \right\}.
\]

**Proof of Theorem 2.** For \(\lambda < 0\) and \(u \in X\), let \(N(\lambda, u)(x) = u''(x) + \lambda u(x) + q(x)g(|u(x)|)u(x)\). Then \(N \in C^1((-\infty, 0) \times X, L^1)\) and, for \(u, v \in X\), \(D_uN(\lambda, u)v = Sv + P(u)v\) where \(Sv = v'' + \lambda v\) and \(P(u)v(x) = q(x)h'(|u(x)|)v(x)\). Since \(\lambda < 0\), the mapping \(S: X \to L^1\) is an isomorphism. Furthermore, for \(u \in X\), we have that \(\lim_{x \to \infty} u(x) = 0\) and hence \(P(u): X \to L^1\) is a compact linear operator.

It follows that \(D_uN(\lambda, u) = S + P(u): X \to L^1\) is an isomorphism if and only if it is injective.

To use the implicit function theorem to prove Theorem 2, we must show that \(D_uN(\lambda, u): X \to L^1\) is injective whenever \((\lambda, u)\) is a solution to \((N)\) with \(\lambda < 0\) and \(u \in K\).

If \((\lambda, u)\) satisfies \((N)\) and \(u \in K\), we have
\[
u''(x) + \lambda u(x) + q(x)h(u(x)) = 0 \quad \text{for all} \quad x > 0 \quad (3.2)
\]
and if \(v \in X \setminus \{0\}\) is such that \(D_uN(\lambda, u)v = 0\), we have
\[
u''(x) + \lambda v(x) + q(x)h'(u(x))v(x) = 0 \quad \text{for all} \quad x > 0. \quad (3.3)
\]
Solutions to a semilinear equation

Hence,
\[ \int_0^\infty q(x)\{h(u(x))v(x) - h'(u(x))v(x)u(x)\} \, dx = 0. \]

Since \( sh'(s) > h(s) \) for \( s > 0 \) and \( u(x) > 0 \) for \( x > 0 \), it follows that there exists \( z > 0 \) such that \( v(z) = 0 \). Furthermore, as in the proof of Lemma 3.1, there exists \( z_1 > 0 \) such that \( v'(x)v(x) < 0 \) for all \( x > z_1 \). Thus, replacing \( v \) by \(-v\) if necessary, we can suppose that \( v(z) = 0, v'(z) > 0 \) and \( v(x) > 0 \) for all \( x > z \). On setting \( w(x) = u'(x) \), we have that \( w(x) < 0 \) for all \( x > 0 \) and
\[
-w''(x) + \lambda w(x) + q(x)h'(u(x))w(x) + q'(x)h(u(x)) = 0, \tag{3.4}
\]
From (3.3) and (3.4), it follows that
\[ -v'(z)w(z) + \int_z^\infty \left[ -v'(x)w'(x) + \lambda v(x)w(x) + q(x)h'(u(x))v(x)w(x) \right] \, dx = 0 \]
and
\[ -w'(z)v(z) + \int_z^\infty \left[ -w'(x)v'(x) + \lambda v(x)w(x) + q(x)h'(u(x))v(x)w(x) \right] \, dx \\
\quad = - \int_z^\infty q'(x)h(u(x))v(x) \, dx. \]
Thus, \( v'(z)w(z) = -\int_z^\infty q'(x)h(u(x))v(x) \, dx \geq 0 \) and so \( v'(z) \geq 0 \). This contradicts the fact that \( v'(z) > 0 \) and we conclude that \( \mathcal{D}_u N(\lambda, u) : X \to L^1 \) must be injective.

In view of Lemmas 3.1 to 3.4, the proof of Theorem 2 is completed by establishing the following fact. A subsequence converging in \( \mathbb{R} \times X \) can be extracted from any sequence \( \{(\lambda_n, u_n)\} \) of solutions to (N) such that
\[ \lambda_n \to \lambda \quad \text{with} \quad \lambda < 0, \]
\[ u_n \in K, \]
\[ |u_n|_\infty \leq \|u_n\|_X \leq C \quad \text{for all} \quad n. \]

To prove this fact, we note first that since \( u_n \in K \),
\[ xu_n(x) \leq \int_0^x u_n(y) \, dy \leq \|u_n\|_X \leq C \quad \text{for all} \quad x > 0. \tag{3.5} \]
By Lemma 3.3(a) and the property (i) of the function \( H \),
\[ -\lambda_n u_n(x)^2 - u_n'(x)^2 \leq 2q(x)H(u_n(x)) \leq 2q(0)u_n(x)^rC^{-r}H(C) \quad \text{where} \quad r > 2. \]
Hence, using (3.5), we obtain
\[ -\lambda_n - u_n'(x)^2/u_n(x)^2 \leq 2q(0)x^{-r+2}C^{-2}H(C). \]
It follows that there exist \( m \) and \( z \) such that
\[ u_n'(x)/u_n(x) \leq -\frac{1}{2}\sqrt{-\lambda} \quad \text{for all} \quad x \geq z \quad \text{and} \quad n \geq m. \]
and consequently,
\[
0 < u_n(x) \leq C \exp \left\{ -\frac{1}{2} \sqrt{-\lambda(x-z)} \right\} \quad \text{for all } x \geq z \text{ and } n \geq m.
\]

By using this estimate and the equation (N), the existence of a subsequence of \( \{u_n\} \) converging in \( X \) is easily established.

This completes the proof of Theorem 2.

**Remark.** From the results stated in [11], it follows that all positive solutions to (N) belong to the branch given by Theorem 2.

**References**


(Issued 12 December 1985)