Pulse Detection With A Multi-State System

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***Abstract*--A multi-state system with potential for pulse detection is proposed. The system is based on utilizing a linear and a quantized feedback path and exhibits integration without the usual droop inherent in practical integrators. The integrator time constant is infinite. This holds provided the output rms noise level, at the three sigma level, is less than half the quantization level used. Effectively, the multi-state system acts as an integrator for a pulse input but acts as a broadband amplifier for the input noise. For pulse detection, it is shown that the performance is similar to that of a matched filter detector for the case of white noise. The structure is such that it enables sub-threshold detection of a pulse train.**

Keywords--multi-state system, quantized feedback, pulse detection, matched filter, white noise.

I. INTRODUCTION

Complex systems are diverse in nature and their modelling is system dependent. Many complex systems are naturally modelled as multi-state systems with their operation being defined by dynamics between states with various feedback paths defining the options for state transitions, e.g. [1]. Cooperative behaviour, as found in self organizing systems, e.g. the decision making model [2], is based on modelling interactions between a collection of entities where local outcomes are dependent on connected state levels.

A useful model for a one-dimensional multi-state system is defined by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \sin[2\pi x(t)] \quad (1)$$

which comprises, as illustrated in Fig. 1, of a sequence of stable states interleaved by metastable states. A similar multi-state system, based on the use of a quantizer, is illustrated in Fig. 2. This paper examines the potential such a multi-state system has for pulse detection when half the quantization level is greater than three times the rms output noise level. For such a case, the multi-state system acts as an integrator for a pulse input but acts as a broadband amplifier for the input noise. A result of this is that there is effective signal integration without the droop inherent in practical integrators and which requires additional circuitry to eliminate [3]. For pulse detection, it is shown that the performance is similar to that of a matched filter detector for the case of white noise. The structure enables sub-threshold detection of a pulse train and automatic counting of distinct pulses in a pulse train.

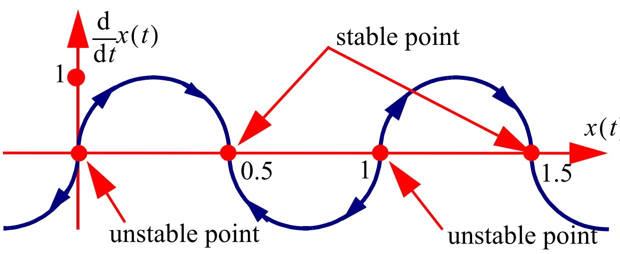

Fig. 1. Relationship between $x(t)$ and $x'(t)$ for the multi-stable system specified by (1).

II. MULTI-STATE SYSTEM

Consider a multi-state system shown in Fig. 2. The differential equation characterizing the system is

$$x^{(1)}(t) = -k_h f_B x(t) + k_h f_B \Delta \left\lfloor \frac{x(t) + 0.5\Delta}{\Delta} \right\rfloor + k_h[s(t) + n(t)] \quad (2)$$

where s is the input signal and n accounts for the noise at the input. The feedback transfer function is illustrated in Fig. 3 and, for the normalized case defined by $\Delta = 1$, there are stable points at $x(t) = \ldots -1, 0, 1, \ldots$.

For small signal variations around a stable point, use of the Laplace transform, for the zero noise case, yields

$$X(s) = \frac{GS(s) + x(0)/p}{1 + s/p}, \qquad G = 1/f_B, p = k_h f_B. \quad (3)$$

Thus, for small signal operation around a stable operating point: gain $= 1/f_B$, $f_{3\mathrm{dB}} = p/2\pi = k_h f_B/2\pi$ (Hz) and the gain-bandwidth product is gbwp $= k_h/2\pi$. Importantly, for an impulsive input with an area of A_p, the expected step change in the output signal is $k_h A_p$.

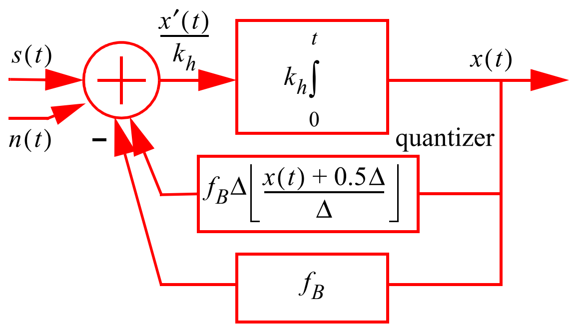

Fig. 2. Quantizer based multi-state system.

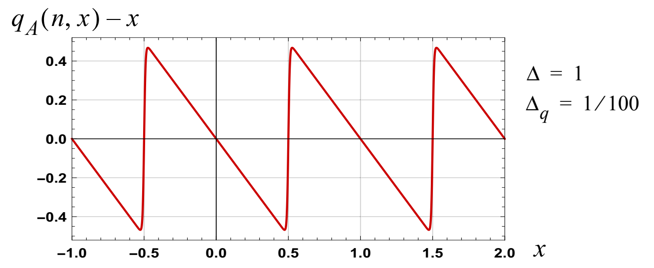

Fig. 3. The feedback transfer function (normalized) as defined by the quantizer transfer function (see (10)) and the linear feedback path.

III. SYSTEM ANALYSIS

Of interest is the response of the system to a pulse input which is of sufficiently short duration such that it acts as an impulse. A half sine pulse defined according to (normalized input amplitude case)

$$s(t) = \sin(2\pi f_o t)[u(t) - u(t-D)], \tag{4}$$

with a duration of $D = 1/2f_o$ and an area of $1/\pi f_o$, is used.

A. System Pulse Response

The response of a first order system with a transfer function, consistent with (3), defined by

$$H(s) = \frac{G}{1+s/p}, \qquad f_{3dB} = \frac{p}{2\pi}, G = \frac{1}{f_B}, \tag{5}$$

to the pulse defined by (4) is

$$x(t) = Gx_1(t)u(t) + Gx_1(t-D)u(t-D) \tag{6}$$

where

$$x_1(t) = \frac{2\pi f_o p}{p^2 + 4\pi^2 f_o^2}\left[e^{-pt} + \frac{p}{2\pi f_o}\sin(2\pi f_o) - \cos(2\pi f_o)\right]. \tag{7}$$

The pulse response is shown in Fig. 4 for the case of $f_o = 1$. A measure of when the system cannot follow the input pulse, i.e. a measure of when the input signal acts as an impulse, is

$$f_{3\text{dB}} \le f_o/2 = 1/4D. \tag{8}$$

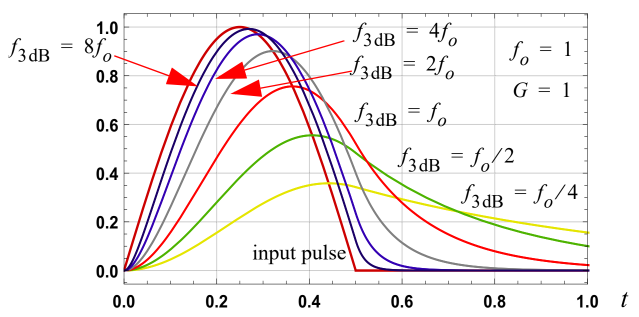

Fig. 4. Pulse response of system as defined by (6).

B. Quantizer Modelling

An ideal quantizer, with a resolution of Δ, has a transfer function defined by

$$q(x) = \Delta\left\lfloor \frac{x + 0.5\Delta}{\Delta} \right\rfloor. \tag{9}$$

To facilitate simulation, the following continuous approximation to the quantizer transfer function, over the range $-n\Delta$ to $n\Delta$, is used:

$$q_A(n, x) = -n\Delta + \sum_{i=-n+1}^{n} \frac{\Delta}{2}\left[1 + \tanh\left(\frac{x + 0.5\Delta - i\Delta}{\Delta_q}\right)\right] \tag{10}$$

where $\Delta_q = \Delta/100$. The overall feedback transfer function is shown in Fig. 3.

C. Noise Characterization

The input noise signal is assumed to be white over the interval $[0, f_{\max}]$, with an average power of A_n^2, and is defined according to [4]

$$n(t) = \sum_{k=1}^{N} \frac{\sqrt{2}A_n}{\sqrt{N}}\sin(2\pi f_k t + \phi_k) \tag{11}$$

where the frequencies $\{f_1, \dots, f_N\}$ are chosen at random from the interval $[0, f_{\max}]$ and $\{\phi_1, \dots, \phi_N\}$ are chosen at random from $[-\pi, \pi]$. The power spectral density of such a noise signal is approximated by

$$G_n(f) \approx \frac{A_n^2}{2f_{\max}}, \qquad -f_{\max} < f < f_{\max}. \tag{12}$$

D. Minimum Detectable Pulse Area

First, an input pulse with a short duration, such that it acts as an impulse, is assumed. Its area is denoted A_p. For this case the output signal level is

$$\text{output signal level} = k_h A_p = 2\pi \cdot \text{gbwp} \cdot A_p \tag{13}$$

Second, the noise variance at the output, for the case of white noise, and small signal operation around a set operating point, is

$$\sigma^2 = \int_{-\infty}^{\infty} G_n(f)|H(f)|^2 df = 2G_n(0)\cdot\left[\frac{\pi}{2}\cdot\frac{k_h f_B}{2\pi}\right]\cdot\frac{1}{f_B^2} \tag{14}$$

where the noise equivalent bandwidth result for a single pole transfer function, e.g. [5], has been used. Thus:

$$\sigma = \sqrt{\pi G_n(0)} \cdot \sqrt{f_{3\mathrm{dB}}} \cdot \mathrm{gain} = \sqrt{\pi G_n(0)} \cdot \frac{\mathrm{gbwp}}{\sqrt{f_{3\mathrm{dB}}}} \quad (15)$$

Consistent with (13) and (15), for gbwp fixed and a fixed output signal level, the output rms noise level increases according to the $\sqrt{\mathrm{gain}}$ and decreases according to $1/\sqrt{f_{3\mathrm{dB}}}$. Hence, improved performance is obtained, for gbwp fixed, by decreasing the gain and increasing $f_{3\mathrm{dB}}$.

The minimum detectable pulse area, defined by the situation where the output level equals the quantization level, and the rms noise level equals the half the quantization noise level divided by three (the three standard deviation case) implies:

$$\begin{cases} 2\pi \cdot \mathrm{gbwp} \cdot A_p = \Delta \\ \sqrt{\pi G_n(0)} \cdot \dfrac{\mathrm{gbwp}}{\sqrt{f_{3\mathrm{dB}}}} \le \dfrac{\Delta}{6} \end{cases} \Rightarrow \quad (16)$$

$$A_p \ge \frac{\Delta}{2\pi \cdot \mathrm{gbwp}} \ge \frac{6}{2\pi \cdot \mathrm{gbwp}} \cdot \left[\sqrt{\pi G_n(0)} \cdot \frac{\mathrm{gbwp}}{\sqrt{f_{3\mathrm{dB}}}}\right]$$

Hence, the minimum detectable pulse area is

$$A_p(\min) = \frac{3}{\sqrt{\pi}}\sqrt{\frac{G_n(0)}{f_{3\mathrm{dB}}}}. \quad (17)$$

For a fixed noise level, and fixed gain-bandwidth product, the minimum detectable pulse area is inversely proportional to the square root of the system bandwidth and can be decreased by increasing the system bandwidth (decreasing the system gain).

However, as the bandwidth is increased, a point is reached where the system no longer acts impulsively with respect to the input pulse. For a pulse duration of D seconds this is specified by (8). Assuming the maximum bandwidth is $f_{3\mathrm{dB}} = 1/4D$, it follows that the minimum detectable pulse area is

$$A_p(\min) = \frac{6}{\sqrt{\pi}}\sqrt{G_n(0)D}. \quad (18)$$

E. Matched Filter Pulse Detection

Consider the case of pulse detection by a matched filter, e.g. [6], where the matched filter impulse response is defined by

$$h(t) = s(T-t) \quad (19)$$

for the case of the input signal s being a pulse. The output of the matched filter detector is

$$x(t) = \int_0^t s(\lambda)h(t-\lambda)d\lambda = \int_0^t s(\lambda)s(T-t+\lambda)d\lambda. \quad (20)$$

At the time $t = T$, the matched filter output is a maximum and equals the pulse energy in the interval $[0, T]$, i.e.

$$x(T) = \int_0^T s^2(\lambda)d\lambda = \text{pulse energy}. \quad (21)$$

For the case of white noise, with a power spectral density of G_n, the noise variance at the filter output is

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} G_n(f)|H(f)|^2 df = G_n(0)\int_0^{\infty}|h(t)|^2 dt \\ &= G_n(0) \cdot \text{pulse energy}\end{aligned} \quad (22)$$

where Parseval's relationship has been used. With the minimum detectable pulse energy being defined as the energy when the output signal level is equal to the rms noise level, it follows that:

$$\begin{aligned}&\text{pulse energy} \ge \sqrt{G_n(0) \cdot \text{pulse energy}} \\ \Rightarrow \quad &\text{pulse energy} \ge G_n(0)\end{aligned} \quad (23)$$

For the case where the signal level is equal to three times (or more) the rms noise level, it follows that

$$\text{pulse energy} \ge 9G_n(0). \quad (24)$$

F. Comparison of Pulse Detection Approaches

Consider the case of a rectangular pulse with a duration of D seconds and a height of A, consistent with a pulse area of AD and a pulse energy of A^2D. For such a case and using the situation of the output level being three times the rms signal level:

$$\begin{aligned}&AD \ge 6/\sqrt{\pi} \cdot \sqrt{G_n(0)D} \qquad \text{multi-state system} \\ &\Rightarrow \qquad A^2D \ge 36/\pi \cdot G_n(0) \\ &A^2D \ge 9G_n(0) \qquad \text{matched filter detection}\end{aligned} \quad (25)$$

Hence, for this case, the multi-state system yields slightly poorer performance, in terms of minimum detectable signal pulse energy, in comparison to a matched filter detector.

IV. RESULTS

Results are presented for the normalized case defined by: $\mathrm{gain} = 1$, $f_{3\mathrm{dB}} = 1$, $\mathrm{gbwp} = 1$ and unit quantization level, $\Delta = 1$. For this case, the gain for the pulse area, as defined by (13), is 2π. The input pulse duration is assumed to be such that the system acts impulsively. Consistent with (8), $D = 1/8f_{3\mathrm{dB}}$ and $f_o = 4f_{3dB}$ is assumed. Consistent with (4) and (13), the input pulse amplitude is $\pi f_o\Delta/k_h = f_o\Delta/2\mathrm{gbwp}$ for an output signal level of Δ.

The input noise is assume to be white over the frequency range $-10f_{3\mathrm{dB}} < f < 10f_{3\mathrm{dB}}$, i.e. $f_{\max} = 10f_{3\mathrm{dB}}$. The input

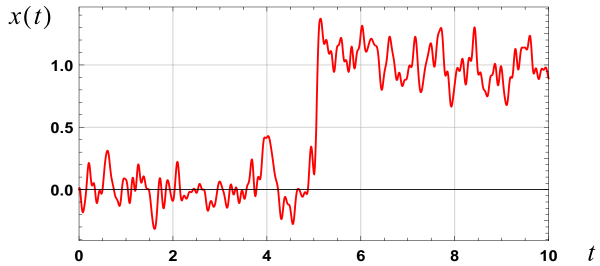

Fig. 5. Detection of a single pulse for the case of an impulsive input at $t = 5$ with an area consistent with an output level of unity. The output rms noise level is equal to $\Delta/6$.

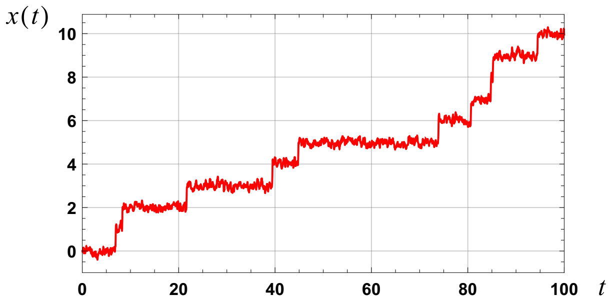

Fig. 6. Detection of the first ten pulses in a pulse train, with a pulse rate of 0.1. The output rms noise level is equal to $\Delta/8$.

rms noise level, for the case of the output rms noise level being equal to $\Delta/6$, then is

$$A_n = \frac{\Delta}{6} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\sqrt{f_{max} \cdot f_{3dB}}}{\text{gbwp}} \quad \textbf{(26)}$$

which follows from (12) and (15). The number of sinusoids comprising each noise signal is assumed to be 1000.

A. *Pulse Detection*

One example of the detection of a single pulse is shown in Fig. 5. The pulse amplitude is $\pi f_o/k_h$ and this level is consistent with an output level of unity.

The detection of a pulse train, with random times as defined by a Poisson point process with a rate of $\lambda = 0.1$, is shown in Fig. 6.

B. *Sub-Threshold Detection*

The detection of a pulse train, which is embedded in a white noise signal, is illustrated in Fig. 7 for the case of pulse amplitudes equal to $0.5\pi f_o/k_h$ and consistent with an individual output pulse level of $\Delta/2$. This level is below the level of detection in the absence of noise.

C. *Notes*

For separated pulses in a pulse train comprising of equal area pulses, the structure is an automatic counter of the number of pulses. In general, the output is consistent with the integral of the input signal but without the droop inherent in a practical integrator. The time constant is infinite. When the input pulse area is not consistent with a stable output level, there is a transient to the closest stable output level. The time constant associated with this transient is defined by the small signal bandwidth and is $\tau = 1/p = 1/k_h f_B$. For the case where the input pulse area does not precisely correspond to one of the quantized output levels, a quantization error in the output exists. A quantizer, based on a sample and hold circuit, yields very similar performance provided the sample rate is greater than the system bandwidth.

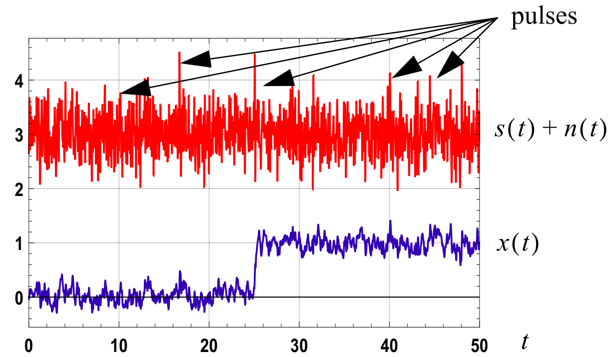

Fig. 7. Top: A pulse train (offset by 3) embedded in a white noise signal with the noise level consistent with an output rms noise level of $\Delta/8$. Bottom: output signal illustrating sub-threshold detection.

V. CONCLUSION

A multi-state system, based on a linear and a quantized feedback path, with potential for pulse detection, has been proposed. Provided half the quantization level is greater than three to four times the rms output noise level, the system acts as an integrator for impulsive input pulses but like a broadband amplifier for the input noise. The result is pulse integration without droop. For pulse detection, it is shown that the performance is similar to that of a matched filter detector for the case of white noise. The structure is such that it enables sub-threshold detection of a pulse train.

VI. REFERENCES

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