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The influence of non-linear elasticity on the determination of Weibull parameters using the fibre bundle tensile test

B. Moser[a,b,*], L. Weber[a], A. Rossoll[a], A. Mortensen[a]

[a]Laboratory for Mechanical Metallurgy, Swiss Federal Institute of Technology EPFL, Lausanne, Switzerland
[b]Swiss Federal Laboratories for Materials Testing and Research EMPA, Thun, Switzerland

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Abstract

We address the influence of individual fibre stress–strain non-linearity on the extraction of Weibull-parameters from fibre bundle tensile tests. We extend the statistical theory of fibre bundle strength to include the non-linear elastic behaviour observed in many technically important fibres, e.g. glass-, carbon-, and alumina-fibres. It is shown that neglecting this non-linearity may lead to significant errors in determining the shape and scale parameters of the fibre fracture strength Weibull-distribution. A refinement of the existing extraction technique, accounting for this effect, is presented. The error resulting from neglecting the non-linear behaviour is assessed through a parametric study of the Weibull parameters for different levels of non-linearity. Explicit calculations are performed for two fibres of technical importance, namely Nextel 610™ α-alumina fibre and a T300 carbon fibre.

Keywords: A. Fibres; A. Tow; C. Statistical properties/methods; D. Mechanical testing; Non-linear elasticity

1. Introduction

Engineering fibres often exhibit a brittle fracture behaviour and their strength is described by a two-parameter Weibull distribution [1]. The probability, S_{Weibull}, that a fibre of length L will sustain an applied stress, σ_f, without breaking is then

$$S_{\mathrm{Weibull}} = \exp\left[-\frac{L}{L_0}\left(\frac{\sigma_f}{\sigma_0}\right)^m\right], \quad (1)$$

where σ_0 is the characteristic fibre strength (often called the scale parameter) of a fibre with length L_0. Parameter m is the Weibull modulus, which describes the strength variability (often referred to as the shape parameter). Eq. (1) describes a size effect on fibre strength: experimental results indicate, however, that the observed size effect does not always follow this Weibull distribution [2]; consequently, some modifications to this equation have been proposed, introducing an additional parameter to account for non-Weibull size effects [3]. Despite these shortcomings, the Weibull distribution, as shown in Eq. (1), has remained the basic equation used to quantify statistical features of the tensile strength of fibres.

The main methods used to determine the statistical parameters, σ_0 and m, of engineering fibres are (i) single fibre tensile tests and (ii) fibre bundle tensile tests [4]. At least 50 single fibre tensile tests are needed to extract these statistical parameters with sufficient precision. These tests are therefore time-consuming and prone to errors caused by a sampling problem: weak fibres are likely to fail during handling prior to testing and are not accounted for in the final strength statistics. This can artificially raise the mean strength and the measured Weibull modulus. Conversely, fibre bending (a prominent occurrence in single fibre tests) can lower the calculated value of both the apparent strength and the Weibull modulus.

In contrast, fibre bundle tensile tests include a greater number of fibres (usually hundreds or thousands) evaluated in a single test. Despite some potential problems (such as errors resulting from misalignment of the fibres within the bundle), such tests are therefore faster and are increasingly used today.

* Corresponding author. Present address: Suresh Group, Massachusetts Institute of Technology, Room 8-139, Cambridge, MA 02139, USA. Fax: +1-617-258-0390.
E-mail address: benedikt.moser@alumni.ethz.ch (B. Moser).

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Following the work of Daniels [5] and Coleman [6] the stress–strain relationship of a fibre bundle can be predicted from the single fibre strength distribution parameters. These studies formed the basis for a number of more refined analyses, where certain initial restrictions were relaxed (e.g. the assumption of a constant fibre length in the bundle or the supposition that the strength distribution follows a Weibull distribution) [7,8]. Several methods for extracting the Weibull parameters from the experimental fibre bundle stress–strain curves have also been developed [9–17]. Some of these are based on the classical fibre bundle theory [9–12] while others use more refined treatments, featuring single fibre strength distributions other than the two-parameter Weibull distribution [13,14], non-constant filament lengths [14] or fibre–fibre interactions [17,18]. These methods have been used to describe the behaviour of numerous fibre materials, including glass [16,19], carbon [9,12,14], Kevlar [17], and ceramic [15]. Despite these improvements, an important point has—to the best knowledge of the authors—never been addressed to date, namely the often non-linear elastic behaviour of high-strength brittle fibres, which becomes noticeable at strains lower than their average fracture strain [20–22]. In the following, we assess the influence of the elastic non-linearity of the fibres on the extraction of the statistical strength parameters from the stress–strain curve of a classical fibre bundle. We first extend the analysis as it is known for linear elastic fibres to non-linear fibre behaviour. The effect is then illustrated using as practical examples a continuous alumina fibre (Nextel 610™) and a carbon fibre (T300).

2. Background: measurement of Weibull parameters from fibre bundle tensile tests

The stress–strain curve and the strength of classical fibre bundles can easily be evaluated under the following assumptions:

1. The number of fibres in the bundle is infinite.
2. All fibres within the bundle have equal cross-sectional area and equal (unit) length, L_0.
3. The load released by the breaking of a fibre is equally distributed over the remaining intact fibres (no interaction between fibre breaks).
4. Each individual fibre has the same probability of failure, which follows the two-parameter Weibull distribution expressed in Eq. (1).
5. The fibre strength is independent of strain-rate.

Under these assumptions, the strength of the fibre bundle is

$$\bar{\sigma}_{max} = \sigma_0 m^{-\frac{1}{m}} \exp\left(-\frac{1}{m}\right), \tag{2}$$

where the stress, $\bar{\sigma}_{max}$, is defined as the applied load divided by the sum of the initial cross-sectional area of all the fibres in the bundle. If the fibre stress–strain relationship is given by $\sigma = f(\varepsilon)$, then the stress–strain curve of the fibre bundle in a strain-controlled tensile test is

$$\bar{\sigma} = f(\varepsilon) \exp\left[-\left(\frac{f(\varepsilon)}{\sigma_0}\right)^m\right]. \tag{3}$$

The strain, ε_{max}, at the fibre bundle strength, $\bar{\sigma}_{max}$, is given by

$$\varepsilon_{max} = f^{-1}\left(\sigma_0 m^{\frac{1}{m}}\right), \tag{4}$$

where f^{-1} is the reciprocal of $f(\varepsilon)$, which we assume to be bijective. Details of this classical analysis can be found in Refs. [9–12].

Numerical studies have shown that Eqs. (2)–(4) can still be relatively accurate when the number of fibres in the bundle is not infinite. McCartney [23] showed that for $m = 8$ a number of 100 fibres or greater is sufficient. This number will be lower for higher Weibull moduli. Since commercial fibre tows typically consist of several hundreds or thousands of filaments this requirement is generally accommodated.

Eq. (3) shows that the theoretical stress–strain curve for an intact fibre bundle, $\sigma = f(\varepsilon)$ and the curve for a fibre bundle with an increasing number of fibre breaks are related through the Weibull parameters only. The Weibull shape parameter, m, and the scale parameter, σ_0, can therefore be conveniently extracted by comparing the two curves at the strain ε_{max}, corresponding to the maximum load. From the bundle stress–strain curve, Weibull parameters are obtained as

$$m = \left[\ln\left(\frac{f(\varepsilon_{max})}{\bar{\sigma}_{max}}\right)\right]^{-1}, \tag{5}$$

and

$$\sigma_0 = f(\varepsilon_{max}) m^{\frac{1}{m}}. \tag{6}$$

In practice, $\bar{\sigma}_{max}$ and ε_{max} are measured from the bundle stress–strain curve, and m and σ_0 are computed from Eqs. (5) and (6).

3. Influence of non-linear fibre behaviour

3.1. General formulation

If fibres are non-linear elastic, all equations in Section 2 remain valid, and the Weibull parameters can still be calculated using Eqs. (5) and (6), provided $f(\varepsilon)$ accounts for the non-linear stress–strain behaviour of the fibres: the value of m calculated by this method depends directly on the fibre stress–strain relation, $f(\varepsilon)$. If a linear stress–strain relation is assumed when extracting the Weibull parameters from a fibre bundle test, significant error may therefore occur if the fibre behaves non-linearly in reality: the Weibull shape parameter, m, will be underestimated (overestimated) for a decreasing (increasing) fibre modulus with increasing strain. The reverse trend is observed for the scale parameter, σ_0, however, the effect of the non-linearity is somewhat less substantial. σ_0 varies roughly linearly with $f(\varepsilon_{\max})$ whereas m has an inverse logarithmic dependence that has a steeper slope in the range of practical interest ($3 < m < 15$).

3.2. Formulation for second order non-linear elasticity

Non-linear elastic deformation of stiff fibres in the longitudinal direction can be described by adding a linear strain-dependent term to the initial (constant) Young's modulus, E_f^0 [24]. The apparent instantaneous fibre Young's modulus, $E_f(\varepsilon)$ is then given as

$$E_f(\varepsilon) = E_f^0(1 + \alpha\varepsilon), \tag{7}$$

where the parameter α, which describes the elastic non-linearity, can range from -7 [25,26] through to 30 [24]. Integration with respect to ε yields the stress–strain function

$$f(\varepsilon) = E_f^0\varepsilon\left[1 + \frac{\alpha}{2}\varepsilon\right]. \tag{8}$$

Weibull parameter extraction errors, resulting from the assumption of linear elastic fibre deformation can now be quantified as a function of α. Fig. 1 shows the variation of the relative error, $1 - m_{\text{linear}}/m$, calculated when linear elasticity is assumed for the evaluation of the experimental data (i.e. taking $\alpha = 0$ to obtain m_{linear}, rather than m). It can be seen that the relative error increases with m and that the error is greater for positive values of the non-linearity parameter α.

The relative error in evaluating m assuming linear elasticity also depends on the characteristic fibre stress σ_0 : it increases for greater characteristic fibre stresses. This dependence is illustrated in Fig. 2 for $\alpha = -10$ (solid lines) and $\alpha = 10$ (broken lines), at various values of m. This, of course, is a direct result of the fact that elastic non-linearity is more pronounced at high strain.

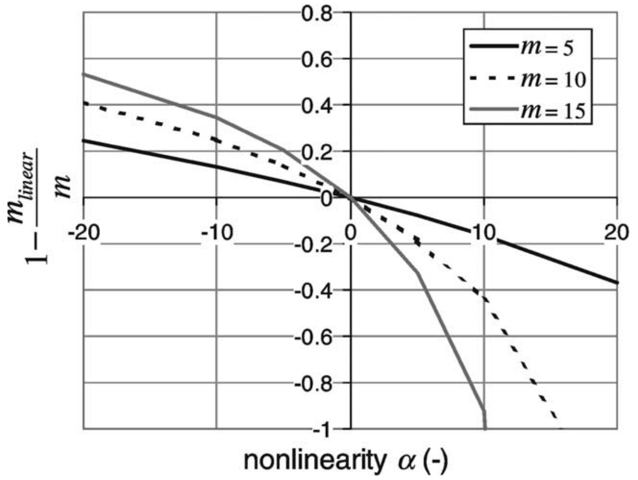

Fig. 1. Relative error in the determination of the shape parameter, m, as a function of the non-linearity parameter, α, for different values of m, with $\sigma_0 = 3$ GPa.

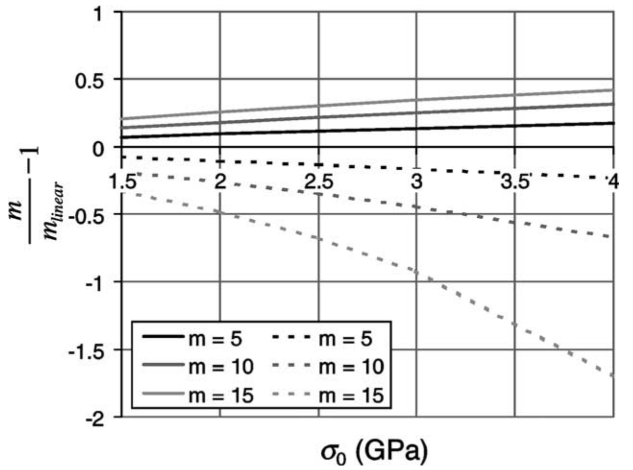

Fig. 2. Relative error in the determination of m_{linear} when using a linear elastic approach for $\alpha = -10$ (solid lines) and $\alpha = 10$ (broken lines) for varying m.

It is seen from Figs. 1 and 2 that, depending on the value of the non-linearity parameter α the error can be substantial. In Section 4, the effect is quantified for two industrial fibres illustrating its importance in practical situations.

4. Application

4.1. Nextel 610™ continuous alumina fibres

The alumina fibre Nextel 610™ (3M, St Paul, MN, USA) is a high strength/high stiffness fibre consisting of fine-grained pure α-alumina. Its general properties are listed in Table 1.

4.1.1. Non-linear elasticity

Elastic non-linearity is a well known phenomenon in engineering ceramics and can be described by higher order elastic constants. Compilations of higher order elasticity

Table 1
Main properties of the Nextel 610™ alumina fibre

Composition [35]	$> 99\%$ α-Al_2O_3 0.2–0.3% SiO_2 0.4–0.7% Fe_2O_3
Mean UTS at $l = 25.4$ mm [36]	3.3 GPa
Weibull modulus [36]	9.7–11.2
Young's modulus [36]	373 GPa
Density [35,36]	3.75–3.9 g cm^{-3}
Diameter [36]	11.98 μm
CTE (100–1100 °C) [37]	8×10^{-6} K^{-1}

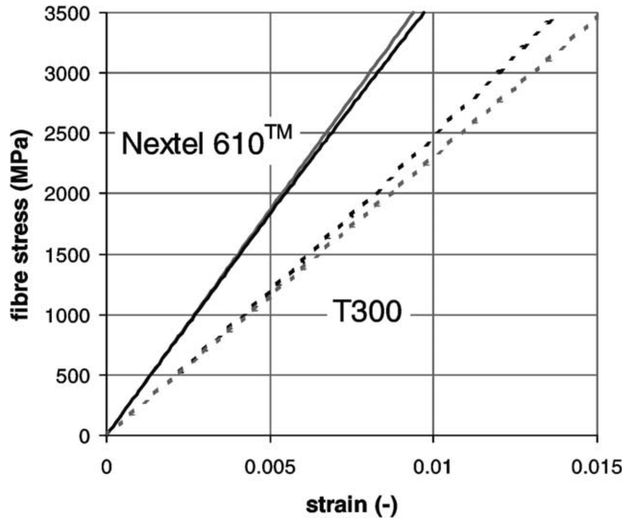

Fig. 3. Stress–strain curve of the Nextel 610™ alumina fibre (solid lines) and the T300 carbon fibre (dashed lines) drawn assuming linear (grey line) and non-linear (black line) elasticity; relevant constants are given in the text.

parameters for single crystal alumina can be found in Simmons and Wang [25] (only pressure derivatives) and Landolt-Börnstein [26]. Higher order elastic constants in compression for polycrystalline alumina are reported in Ref. [27].

In a separate study, we determined the strain dependence of the Young's modulus for Nextel 610™ by measuring the longitudinal modulus strain dependence of aluminium matrix composites, reinforced unidirectionally with 60 vol.% Nextel 610™ alumina fibres [28,29]. A linear decrease of the fibre Young's modulus with increasing strain was observed. The value of α was found to be -6.81 for the α-alumina fibre Nextel 610™. The fibre stress–strain curves assuming linear (grey solid line) and non-linear (black solid line) elasticity are plotted using these parameters in Fig. 3.

4.1.2. The influence on the Weibull parameters

When the non-linear bundle stress–strain curve (calculated according to Eqs. (3) and (8) with the fibre properties from Table 1, taking $m = 11.2$ and $\alpha = -6.81$) is evaluated using a linear elastic approach for the theoretical bundle stress at maximum strain, $f(\varepsilon_{max})$, the result for the Weibull modulus, m_{linear}, is only 8.7; the correct value of 11.2 is significantly underestimated. It is clear that, while the fibre Young's modulus is overestimated by only 20 GPa at $f(\varepsilon_{max})$, the extracted shape parameter is significantly underestimated when non-linear behaviour is neglected in the analysis.

4.2. Carbon fibres

According to the literature, the stiffness of carbon fibres generally increases with increasing strain [21,24, 30–32]. This phenomenon was found to be entirely reversible and is attributed to orientation of the lamellar crystallites. The effect is sufficiently strong to cause stiffening of an aluminium composite, reinforced with a high modulus carbon fibre M40, even though the matrix plastifies and its contribution to the composite modulus decreases [33].

Similar to the non-linearity in the alumina fibre mentioned above, the elastic behaviour of such carbon fibres can be described by second order elasticity, according to Eq. (7), and α-values for several carbon fibres can be found in Ref. [24]. Taking the carbon fibre T300 as an example, the values of α and σ_0 are found to be 15 and 3.5 GPa, respectively. The initial fibre Young's modulus is approximately 230 GPa and the Weibull modulus is about 5 [12,34]. The corresponding stress–strain curves of the individual fibre are illustrated in Fig. 3 (dashed lines). The extraction of the Weibull modulus from the non-linear fibre bundle stress–strain curve with a linear elastic approach yields $m_{linear} = 7.9$, an overestimation of the actual value by a factor of almost 1.6.

4.3. Experimental uncertainty

In practice, there are a number of additional uncertainties involved with the evaluation of these statistical parameters. These include:

- *Determination of strain at maximum stress*. The strain corresponding to the maximum stress is—as shown above—an important input to the determination of m and σ_0. Taking values from the Nextel 610™ alumina fibre, a relative error of 1% in ε_{max} results in an error in m of approximately 11%, but only a 2.5% error is observed for σ_0.
- *The number of filaments in the fibre bundle and their average diameter*. These parameters affect the analysis either through the calculation of the average fibre bundle stress or through the calculation of the theoretical fibre bundle load (when load instead of stress values are used). Again, for the Nextel 610™ alumina fibre a relative error of 1% in $\bar{\sigma}_{max}$ results in an error of about 11% in m, but only a 1.5% error in σ_0. It is important to note that if the number of filaments in the fibre bundle is calculated from the initial stiffness of the bundle (assuming no fibre breaks at this stage), an error in the strain measurement will result in the same relative error in the stress measurement (through the cross-sectional area), with no effect on the calculation of the Weibull shape parameter; the two errors compensate each other. There will, however, still be an error in the calculation of the scale parameter, since this is a linear function of ε_{max}, Eq. (6).
- *Unequal fibre length in the bundle*. Depending on the overall length of the fibre bundle tested, a broader or

narrower distribution of fibre lengths will be present. This fibre slack will result in an initial stiffening of the bundle upon loading and in an inhomogeneous stress distribution when all the slack is taken up. As previously mentioned, the Weibull parameters can still be calculated [3,14], however, the measurement of the initial stiffness (used to determine the number of filaments in the bundle) might not be possible depending on the amount of this slack.

5. Conclusions

The elastic behaviour of engineering fibres can be noticeably non-linear, as evidenced by the preceding ceramic and carbon fibre examples. Neglecting this effect results in significant error when the fibre strength Weibull modulus is computed using the maximum stress, and its corresponding strain, measured in a bundle test. The Weibull modulus is underestimated when the deviation from linear elasticity is negative (e.g. ceramic fibres) and is overestimated when this deviation is positive (e.g. carbon fibres). Modified expressions for the computation of fibre strength shape and scale parameters are given for fibres having a non-linear elastic behaviour characterised by Eq. (8). This, and other apparently minor sources of experimental errors can have a significant influence on extracted Weibull parameters and must therefore be carefully evaluated when using fibre bundle testing to assess fibre Weibull statistics.

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