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Generalized differential Galois theory. (English summary)
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A generalization of differential Galois theory using two sets of mutually commuting differentials $E=\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m\}$ and $\Delta=\{\delta_1,\delta_2,\dots,\delta_n\}$ is derived. The paper shows that all connected differential algebraic groups are Galois groups of some appropriate differential field extension. It takes advantage of the fact that constants with respect to E may not be constants with respect to Δ. The theory presented includes infinite-dimensional groups.

In the non-differential case, if K is a finite normal separable extension of k, then there is a one-to-one correspondence between the subgroups of H of $G(K/k)$ and the subfields L of K which contain k, the corresponding elements H and L being such that L is the fixed field of H and $H=G(K/L)$. Finding the right generalization in the differential setting is far from easy, mainly because splitting fields of differential polynomials are missing.

One of the first differential Galois theories handled solutions of a homogeneous linear ordinary differential equation. A differential field extension $\mathcal{G}$ of the differential field $\mathcal{F}$ of characteristic 0 with algebraic closed field of constants is a Picard-Vessiot extension if $\mathcal{G}=\mathcal{F}\langle\eta_1,\dots,\eta_n\rangle$, where $\eta_1,\dots,\eta_n$ constitute a fundamental system of solutions of the homogeneous differential polynomial $\sum_{i=0}^n p_i y^{(n-i)}$ (with $p_0=1$ and $p_i\in\mathcal{F}$, a solution meaning that the polynomial vanishes upon replacement of y by $\eta_i, i=1,\dots,n$). Both $\mathcal{G}$ and $\mathcal{F}$ share the same field of constants. E. R. Kolchin clarified many terms, such as transcendental, integral, exponential, and Liouvillian solutions, by using a precise distinction when applied to the necessary and sufficient part of the Picard-Vessiot theory available at that time. He established key theorems purely algebraically, not resorting to analytic function theory [Ann. of Math. (2) **49** (1948), 1–42; MR0024884 (9,561c)]. Indeed, the group-theoretic aspects of Picard-Vessiot theory are best studied through matrix groups instead of Lie groups so as to allow a purely algebraic interpretation without any analytical aspects. A matrix is seen as a point in n^2-dimensional space and the set of matrices forms an algebraic manifold. A triangular decomposition is possible, much like in Lie theory, and is valid for fields of arbitrary characteristic. This affords a sound footing for the algebraic part of differential Galois theory by putting into correspondence an intermediate differential field and a set of subgroups of the matrix groups. However, the characterization of this set of matrix subgroups still remained to be achieved.

In 1953, Kolchin clarified to a better extent the sort of correspondence occurring, but this needed both an adequate definition of an algebraic group and the notion of strongly normal extensions. An $\mathcal{F}$-morphism $\sigma_1\colon\mathcal{F}\to\mathcal{G}$ is strong over σ_2 if $\sigma_1(c)=\sigma_2(c)$ for every constant c of $\mathcal{F}$ and the composite field $\sigma_1(\mathcal{F})\cdot\sigma_2(\mathcal{F})$ is generated over $\sigma_2(\mathcal{F})$ by constants. To obtain a group G, each strong isomorphism of $\mathcal{G}$ over $\mathcal{F}$ is identified with the automorphism of $\mathcal{G}\langle\mathcal{C}\rangle$ over $\mathcal{F}\langle\mathcal{C}\rangle$ where $\mathcal{C}$ stands for the field of constants of the universal field. A differential extension is called strongly normal if for every differential field extension $\mathcal{G}$ of $\mathcal{F}$ and for two arbitrary $\mathcal{F}$-morphisms

$\sigma_1, \sigma_2 \colon \mathcal{F} \to \mathcal{G}$ of differential fields, σ_1 is strong over σ_2.

Thus the main result at that time was if $\mathcal{G}$ is strongly normal over $\mathcal{F}$, then there is a one-to-one mapping between the set of differential fields between $\mathcal{F}$ and $\mathcal{G}$ and the set of algebraic groups in G whose dimension agrees with the differential degree of the extension. However, even when G is taken as the group of all automorphisms of $\mathcal{G}$ over $\mathcal{F}$ and $G(\mathcal{F}_1)$ is a normal subgroup of G, the factor group $G/G(\mathcal{F}_1)$ need not be isomorphic to the group of all automorphisms of $\mathcal{F}_1$. An example is given on p. 793 in [E. R. Kolchin, Amer. J. Math. **75** (1953), 753–824; MR0058591 (15,394a)]. The subgroups still do not constitute a satisfactory algebraic subgroup. The Galois groups are birationally isomorphic to algebraic groups, but not canonically. Thus, one approach is to resort to axiomatizing the notion of algebraic group so as to allow the Galois groups to be an algebraic group. This is similar to the usual Galois theory, where the Galois group is initially a group of permutations; after the axiomatization of the notion of group the Galois groups became groups in their own right. Using the axiomatic treatment, Kolchin introduced a Galois theory using a single set of mutually commuting differentials together with a suitable differential group theory, both of which can be found in his books [*Differential algebra and algebraic groups*, Academic Press, New York, 1973; MR0568864 (58 #27929)] and [*Differential algebraic groups*, Academic Press, Orlando, FL, 1985; MR0776230 (87i:12016)].

The exposition of the paper under review follows a similar organization as Kolchin's axiomatic treatment. Many proofs rely on differential specializations, which can be found in [P. J. Cassidy, Amer. J. Math. **94** (1972), 891–954; MR0360611 (50 #13058)] and [E. R. Kolchin, op. cit.; MR0776230 (87i:12016)]. Let $\mathcal{G}$ be an E-strongly normal extension of $\mathcal{F}$ with Δ-constants $\mathcal{C}$, meaning that $\mathcal{G}$ is an (E,Δ)-generated extension of $\mathcal{F}$ such that every (E,Δ)-$\mathcal{F}$-isomorphism σ of $\mathcal{G}$ is E-strong: this means that (i) σ leaves invariant every element of $\mathcal{G}^\Delta$ (i.e. $\mathcal{G}^\Delta = \mathcal{C}$ is the field of constants of $\mathcal{G}$ with respect to the operators of Δ), (ii) $\sigma\mathcal{G} \subset \mathcal{G}\cdot\mathcal{U}^\Delta$ (where $\cdot$ denotes the compositum and $\mathcal{U}^\Delta$ the field of constants with respect to Δ of the universal field $\mathcal{U}$) and (iii) $\mathcal{G} \subset \sigma\mathcal{G}\cdot\mathcal{U}^\Delta$. Through suitable differential specialization properties, the set of all E-strong (E,Δ)-isomorphisms of $\mathcal{G}$ is canonically identified with the set of all (E,Δ)-automorphisms of $\mathcal{G}\cdot\mathcal{U}^\Delta$. Composition of such automorphisms gives the group operation. This then leads to a Galois correspondence; namely, if $\mathcal{C}$ is constrainedly closed as an E-field, then there is a bijective correspondence between (E,Δ)-subfields $\mathcal{F}_1$ with $\mathcal{F}_1 \subset \mathcal{G}$ and E-subgroups $G_1 \subseteq G(\mathcal{G}/\mathcal{F})$.

Another important result is that a connected differential algebraic group is shown to be the Galois group of a class of strongly normal extensions. First, it is possible to associate to any local derivation χ at $g \in G$ a unique element $l_\chi(g)$ with the property $l_\chi(g)f(g) = \chi(f(g))$ for every E-$\mathcal{F}$ function defined at g. This defines the logarithmic derivative operators $l_{\delta_1}, l_{\delta_2}, \ldots, l_{\delta_n}$ associated with the operators of Δ. Then, let G be a connected E-$\mathcal{C}$-group (relative to the E-field $\mathcal{U}^\Delta$). From G, one constructs $G_{E,\Delta}$ in the following way: (i) Let $G_\mathcal{U}$ be the E-$\mathcal{C}$-group obtained from G (relative to $\mathcal{U}^\Delta$) by extending the universal differential field from $\mathcal{U}^\Delta$ to $\mathcal{U}$; (ii) $G_{E,\Delta}$ is then obtained from the E-$\mathcal{C}$-group $G_\mathcal{U}$ by extending the derivations from E to (E,Δ). Now, the author shows that after choosing any $\mathcal{C}$-generic element η of $G_{E,\Delta}$, the extension $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ is E-strongly normal over $\mathcal{F} = \mathcal{C}\langle l\delta_1\eta\rangle_{E,\Delta}\cdots\mathcal{C}\langle l\delta_m\eta\rangle_{E,\Delta}$ (relative to the (E,Δ)-field $\mathcal{U}$) with the property that the Galois group $G(\mathcal{G}/\mathcal{F})$ (relative to the E-field $\mathcal{U}^\Delta$) is E-$\mathcal{C}$-isomorphic to G. This strengthens the inverse problem of Galois theory given successively by A. Białynicki-Birula

[Proc. Amer. Math. Soc. **15** (1964), 960–964; MR0167487 (29 #4760)] and J. Kovacic [Trans. Amer. Math. Soc. **207** (1975), 375–390; MR0379452 (52 #357)].

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