EFFECTIVE MODELS FOR LONG TIME WAVE PROPAGATION IN LOCALLY PERIODIC MEDIA

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Abstract. A family of effective equations for the wave equation in locally periodic media over long time is derived. In particular, explicit formulas for the effective tensors are provided. To validate the derivation, an a priori error estimate between the effective solutions and the original wave is proved. As the dependence of the estimate on the domain is explicit, the result holds in arbitrarily large periodic hypercube. This constitutes the first analysis for the description of long time effects for the wave equation in locally periodic media. Thanks to this result, the long time a priori error analysis of the numerical homogenization method presented in [*A. Abdulle and T. Pouchon*, *SIAM J. Numer. Anal.*, 54, 2016, pp. 1507–1534] is generalized to the case of a locally periodic tensor.

Key words. homogenization, effective equations, wave equation, heterogeneous media, long time behavior, dispersive waves, a priori error analysis, multiscale method

AMS subject classifications. 35B27, 74Q10, 74Q15, 35L05, (65M60, 65N30)

1. Introduction. The wave equation in heterogeneous media is used to model diverse multiscale applications in engineering such as seismic inversion, medical imaging or the manufacture of composite materials. In such situations, the medium is described by a tensor a^ε, where $\varepsilon > 0$ denotes the characteristic length of the spatial variation of a^ε and is assumed to be much smaller than the wavelength of the initial data and the source term ($\varepsilon \ll 1$). The displacement of the wave $u^\varepsilon : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is then characterized by the equation

$$\partial_t^2 u^\varepsilon(t,x) - \nabla_x \cdot \big(a^\varepsilon(x)\nabla_x u^\varepsilon(t,x)\big) = f(t,x) \quad \text{in } (0,T] \times \mathbb{R}^d, \tag{1.1}$$

where initial conditions for $u^\varepsilon(0,x)$ and $\partial_t u^\varepsilon(0,x)$ are given. Before discretizing (1.1), we truncate the space $\mathbb{R}^d$ to a sufficiently large hypercube Ω, so that the waves do not reach the boundary, and impose periodic boundary conditions (Ω is called a pseudoinfinite domain). To approximate (1.1) accurately, standard numerical methods such as the finite element (FE) method or the finite difference (FD) method require a grid that resolves the whole domain at the microscopic scale $\mathcal{O}(\varepsilon)$. Hence, as T increases (i.e. Ω increases) or as $\varepsilon \to 0$, such methods have a prohibitive computational cost. Therefore, more sophisticated numerical methods are needed.

The study of multiscale problems such as (1.1) is tied to homogenization theory (see [17, 42, 16, 35, 23, 38]). The general homogenization result for the wave equation in [19] provides the existence of a function u^0 such that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ converges weakly in $\mathrm{L}^\infty(0,T;\mathrm{W}_{\mathrm{per}}(\Omega))$ to u^0 as $\varepsilon \to 0$ (see below for the definitions of the functional spaces). The homogenized solution u^0 is characterized by the homogenized equation

$$\partial_t^2 u^0(t,x) - \nabla_x \cdot \big(a^0(x)\nabla_x u^0(t,x)\big) = f(t,x) \quad \text{in } (0,T] \times \Omega, \tag{1.2}$$

where the initial conditions are the same as for u^ε. As the homogenized tensor a^0 in (1.2) is obtained as the so called G-limit of the sequence $\{a^\varepsilon\}_{\varepsilon>0}$ (see [44, 24]), (1.2) does not depend on the microscopic scale and is thus a good target for numerical methods. However, for a general tensor a^ε, a^0 might not be unique and no formula is available for its computation. Nevertheless, when the medium is locally periodic, i.e., $a^\varepsilon(x) = a\big(x, \frac{x}{\varepsilon}\big)$ with $y \mapsto a(x,y)$ Y-periodic, such formula exists. Indeed, in this case $a^0(x)$ can be computed at each $x \in \Omega$ via the solutions of d cell problems, which are elliptic partial differential equations (PDEs) in Y (see e.g. [10, 22]).

In the past few years, several multiscale methods for the approximation of (1.1) have been developed. The physical origin of (1.1) motivates the choice of an appropriate method. In particular, the problems are divided in two classes, depending whether the medium has, or not, scale separation. Let us first mention the methods available if the medium does not have scale separation. We refer to [6] for a detailed review.

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The methods defined in [39], [34, 33], [40] and [7] rely on multiscale FE spaces that have the same number of degrees of freedom (DOF) as in a coarse FE method. However, the construction of these spaces involves the solutions of global elliptic PDEs at the fine scale, which is computationally expensive and might be prohibitive. To settle this issue, the elliptic PDEs are localized to small patches covering the domain, leading to a process that can be parallelized. Let us then introduce the methods available when the medium has scale separation. In such media, numerical methods can take advantage of the specific structure to reduce the computational cost. To that purpose, the heterogeneous multiscale method (HMM) provides an appropriate framework (see [2]). In the HMM, the effective datum is approximated with a sampling strategy by solving local micro problems and is then used at the macro scale with a chosen numerical method. As the micro scale is resolved only locally in small domains, the cost of the HMM is proportional to the number of DOF at the macro scale. Furthermore, as the micro problems are independent, the sampling procedure can be efficiently parallelized. Two HMMs are available to approximate (1.1). The FD-HMM, defined in [28] and analyzed in [13], relies on a FD method at the macro scale. The effective flux is approximated by solving micro problems in space-time sampling domains of size $\tau \times \eta^d$, where $\tau, \eta \geq \varepsilon$. The FE-HMM, defined and analyzed in [3], relies on the FE method on a macro mesh to approximate the homogenized solution. The homogenized tensor is approximated at the quadrature points by solving micro problems in spatial sampling domains of size δ^d, where $\delta \geq \varepsilon$. In the case of a locally periodic tensor, the FD-HMM and the FE-HMM are proved to converge to the homogenized solution u^0.

When considering large timescales $T = \mathcal{O}(\varepsilon^{-2})$, u^ε develops macroscopic dispersive effects. As the homogenized solution does not describe these effects, new numerical methods are needed for the long time approximation of (1.1). In particular, we look for a new effective equation that captures the dispersion. In the literature, several papers [43, 32, 31, 36, 25, 26, 9, 11, 8] investigated the research of long time effective equations in the case of a uniformly periodic tensor, i.e., $a^\varepsilon(x) = a\big(\frac{x}{\varepsilon}\big)$ with $a(y)$ Y-periodic. In particular, a recent result in [8] defines a family of effective equations whose elements are proved to approximate u^ε at large timescales $\mathcal{O}(\varepsilon^{-2})$. The family is composed of equations of the form (we use the convention that repeated indices are summed)

$$\partial_t^2 \tilde{u}(t,x) - a^0_{ij}\partial^2_{ij}\tilde{u}(t,x) + \varepsilon^2\big(a^2_{ijkl}\partial^4_{ijkl}\tilde{u}(t,x) - b^2_{ij}\partial^2_{ij}\partial^2_t\tilde{u}(t,x)\big) = f(t,x) \quad \text{in } (0,\varepsilon^{-2}T]\times\Omega, \tag{1.3}$$

with the same initial conditions as for u^ε, where a^0 is the homogenized tensor (constant in the uniformly periodic case) and a^2, b^2 are non-negative tensors that satisfy a given constraint.

The two HMMs described above have been adapted to the long time approximation of (1.1). In [29], a modification of the FD-HMM is built to capture the effective flux of an ill-posed effective equation derived in [43]. However, to do so, the space-time sampling strategy requires larger sampling domains as $\varepsilon \to 0$. Furthermore, as it is build on an ill-posed model, a regularization step has to be performed. Nevertheless, in one dimension and for uniformly periodic tensors, the method is shown in [12] to capture the effective flux of the ill-posed model. In [5, 4], the FE-HMM was also generalized for long time approximation. The method, called the FE-HMM-L, was analyzed over long time in [9]. In particular, in one dimension and for uniformly periodic tensors, the method is proved to converge to an effective equation of the family (1.3).

In this paper, we generalize the family of effective equations from [8] to the case of a locally periodic tensor. This analysis constitutes the first result in the study of long time wave propagation in locally periodic media. The family consists of equations of the form

$$\partial_t^2\tilde{u}(t,x) - \partial_i\big(a^0_{ij}(x)\partial_j\tilde{u}(t,x)\big) + \varepsilon L^1\tilde{u}(t,x) + \varepsilon^2 L^2\tilde{u}(t,x) = f(t,x) \quad \text{in } (0,\varepsilon^{-2}T]\times\Omega, \tag{1.4}$$

with the same initial conditions as u^ε and where the operators L^1 and L^2 are given as

$$L^1 = -\partial_i\big(a^{12}_{ij}(x)\partial_j\,\cdot\,\big) + b^{10}\partial_t^2, \quad L^2 = \partial^2_{ij}\big(a^{24}_{ijkl}(x)\partial^2_{kl}\,\cdot\,\big) - \partial_i\big(b^{22}_{ij}(x)\partial_j\partial_t^2\,\cdot\,\big) - \partial_i\big(a^{22}_{ij}(x)\partial_j\,\cdot\,\big) + b^{20}\partial_t^2.$$

The tensors a^{mn}, b^{mn} are defined for all $x \in \Omega$ via the solutions of local cell problems and are linked by a parameter. The main result of the paper ensures that any effective solution of the family (1.4) satisfies the error estimate

$$\|u^\varepsilon - \tilde{u}\|_{\mathrm{L}^\infty(0,\varepsilon^{-2}T;W)} \leq C\varepsilon, \tag{1.5}$$

where the norm $\|\cdot\|_W$ is defined in (1.8) and is equivalent to the L^2 norm through the Poincaré constant. As we track the dependency of the estimate on Ω, the result is valid in arbitrarily large hypercubes. Thanks to this result, we prove that, in the one-dimensional case, the FE-HMM-L converges to an effective solution in the locally periodic case. In particular, the approximation u_H satisfies the error estimate

$$\|u^\varepsilon - u_H\|_{\mathrm{L}^\infty(0,\varepsilon^{-2}T;W)} \le C\big(\varepsilon + (h/\varepsilon^2)^2 + H^\ell/\varepsilon^2\big), \tag{1.6}$$

where h is the micro mesh size, H is the macro mesh size, and ℓ is the macro FE degree. Note that, in the last two terms, a factor ε^{-2} comes from the timescale $\mathcal{O}(\varepsilon^{-2})$. We emphasize that thanks to a new elliptic projection, (1.6) can be used in arbitrarily large domain Ω. This result generalizes the long time a priori error analysis of the FE-HMM-L performed in [9].

The paper is organized as follows. First, in Section 2, we present our main result: we define the family of effective equations (1.4) and state the corresponding error estimate. Then, the derivation of the family and the construction of the adaptation are presented in Section 3 and the proof of the main result is performed in Section 4. In Section 5, we provide long time a priori error estimates for the FE-HMM-L in the locally periodic case. Finally, we illustrate our theoretical results in numerical experiments in Section 6.

Definitions and notation. Let us give some definitions and the notation used in the paper. The derivative with respect to the i-th space variable x_i is denoted ∂_i and the derivation with respect to any other variable is specified. We denote the quotient space $\mathcal{L}^2(\Omega) = \mathrm{L}^2(\Omega)/\mathbb{R}$ and a bracket $[v]$ is used to denote the equivalence class of $v \in \mathrm{L}^2(\Omega)$ in $\mathcal{L}^2(\Omega)$. Equipped with the inner product

$$\big([v],[w]\big)_{\mathcal{L}^2(\Omega)} = \big(v - \langle v\rangle_\Omega, w - \langle w\rangle_\Omega\big)_{\mathrm{L}^2(\Omega)} = \big(v,w\big)_{\mathrm{L}^2(\Omega)} - |\Omega|\langle v\rangle_\Omega\langle w\rangle_\Omega \quad \forall v,w \in \mathrm{L}^2(\Omega),$$

$\mathcal{L}^2(\Omega)$ is a Hilbert space. Furthermore, we denote $\mathcal{W}_{\mathrm{per}}(\Omega) = \mathrm{H}^1_{\mathrm{per}}(\Omega)/\mathbb{R}$ and a bold face letter $\boldsymbol{v}$ is used to denote the elements of $\mathcal{W}_{\mathrm{per}}(\Omega)$. The space $\mathrm{W}_{\mathrm{per}}(\Omega)$ (resp. $\mathrm{L}^2_0(\Omega)$) is composed of the zero mean representatives of the equivalence classes in $\mathcal{W}_{\mathrm{per}}(\Omega)$ (resp. $\mathcal{L}^2(\Omega)$). We define the following norm on $\mathcal{W}_{\mathrm{per}}(\Omega)$

$$\|\boldsymbol{w}\|_{\mathcal{W}} = \inf_{\substack{\boldsymbol{w}=\boldsymbol{w}_1+\boldsymbol{w}_2\\ \boldsymbol{w}_i=[w_i]\in\mathcal{W}_{\mathrm{per}}(\Omega)}} \Big\{\|[w_1]\|_{\mathcal{L}^2(\Omega)} + \|\nabla w_2\|_{\mathrm{L}^2(\Omega)}\Big\} \quad \forall \boldsymbol{w} \in \mathcal{W}_{\mathrm{per}}(\Omega), \tag{1.7}$$

and the corresponding norm on $\mathrm{W}_{\mathrm{per}}(\Omega)$

$$\|w\|_W = \inf_{\substack{w=w_1+w_2\\ w_1,w_2\in\mathrm{W}_{\mathrm{per}}(\Omega)}} \Big\{\|w_1\|_{\mathrm{L}^2(\Omega)} + \|\nabla w_2\|_{\mathrm{L}^2(\Omega)}\Big\} \quad \forall w \in \mathrm{W}_{\mathrm{per}}(\Omega). \tag{1.8}$$

We verify that a function $w \in \mathrm{W}_{\mathrm{per}}(\Omega)$ satisfies $\|w\|_W = \|[w]\|_{\mathcal{W}}$. Furthermore, using the Poincaré–Wirtinger inequality, we verify that $\|\cdot\|_W$ is equivalent to the L^2 norm (C_Ω is the Poincaré constant)

$$\|w\|_W \le \|w\|_{\mathrm{L}^2(\Omega)} \le \max\{1, C_\Omega\}\|w\|_W \quad \forall w \in \mathrm{W}_{\mathrm{per}}(\Omega). \tag{1.9}$$

We denote $\mathrm{Ten}^n(\mathbb{R}^d)$ the vector space of tensors of order n. In the whole text, we drop the notation of the sum symbol for the dot product between two tensors and use the convention that the repeated indices are summed. The subspace of $\mathrm{Ten}^n(\mathbb{R}^d)$ of symmetric tensors is denoted $\mathrm{Sym}^n(\mathbb{R}^d)$, i.e., $q \in \mathrm{Sym}^n(\mathbb{R}^d)$ satisfies $q_{i_1\cdots i_n} = q_{i_{\sigma(1)}\cdots i_{\sigma(n)}}$ for any permutation σ. Let $S^n : \mathrm{Ten}^n(\mathbb{R}^d) \to \mathrm{Sym}^n(\mathbb{R}^d)$ be the symmetrization operator defined as $\big(S^n(q)\big)_{i_1\cdots i_n} = \frac{1}{n!}\sum_{\sigma\in S_n} q_{i_{\sigma(1)}\cdots i_{\sigma(n)}}$. In the text, $\big(S^n(q)\big)_{i_1\cdots i_n}$ is denoted as $S^n_{i_1\cdots i_n}\{q_{i_1\cdots i_n}\}$. For $q \in \mathrm{Ten}^4(\mathbb{R}^d)$ and $\xi, \eta \in \mathrm{Sym}^2(\mathbb{R}^d)$, we denote the product $q\xi : \eta = q_{ijkl}\xi_{ij}\eta_{kl}$. A tensor $q \in \mathrm{Ten}^4(\mathbb{R}^d)$ is *major symmetric* if $q_{ijkl} = q_{klij}$ for all $1 \le i,j,k,l \le d$ and it is *positive semidefinite* if $q\xi : \xi \ge 0$ for all $\xi \in \mathrm{Sym}^2(\mathbb{R}^d)$. For a major symmetric tensor $q \in \mathrm{Ten}^4(\mathbb{R}^d)$, there exists a bijective map $\nu : \mathrm{Sym}^2(\mathbb{R}^d) \to \mathbb{R}^{N(d)}$, where $N(d) = \binom{d+1}{2}$, and a matrix $M(q) \in \mathrm{Sym}^2(\mathbb{R}^{N(d)})$ such that (see e.g. [8] or [41, Chapter 4])

$$q\xi : \eta = M(q)\nu(\xi)\cdot\nu(\eta) \quad \forall \xi,\eta \in \mathrm{Sym}^2(\mathbb{R}^d). \tag{1.10}$$

In particular, q is positive (semi)definite if and only if $M(q)$ is positive (semi)definite.

Settings of the problem. We assume that $d \leq 3$ (note that the main result holds for $d > 3$, provided higher regularity assumption of the tensor). Let $a^\varepsilon(x) = a\big(x, \frac{x}{\varepsilon}\big)$ be a $d \times d$ symmetric locally periodic tensor, i.e., $a(x,y)$ is Y-periodic in y and Ω-periodic in x, where $\Omega, Y \in \mathbb{R}^d$ are open hypercubes. We assume that Ω is a union of cells of volume $\varepsilon|Y|$. More precisely, letting $\ell \in \mathbb{R}^d$ be the period of $y \mapsto a(x,y)$ (i.e., $a(x, y + k \cdot \ell) = a(x,y)$ for all $(x,y) \in \Omega \times Y$ and $k \in \mathbb{Z}^d$), we assume that $\Omega = (\omega_1^l, \omega_1^r) \times \cdots \times (\omega_d^l, \omega_d^r)$ satisfies

$$\frac{\omega_i^r - \omega_i^l}{\varepsilon \ell_i} \in \mathbb{N}_{>0} \quad \forall i = 1, \dots, d. \tag{1.11}$$

This assumption ensures that for any Y-periodic function γ, the map $x \mapsto \gamma\big(\frac{x}{\varepsilon}\big)$ is Ω-periodic (γ is extended to $\mathbb{R}^d$ by periodicity). For $T^\varepsilon = \varepsilon^{-2} T$, we consider the wave equation: $u^\varepsilon : [0, T^\varepsilon] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\partial_t^2 u^\varepsilon(t,x) - \nabla_x \cdot \big(a\big(x, \tfrac{x}{\varepsilon}\big)\nabla_x u^\varepsilon(t,x)\big) = f(t,x) && \text{in } (0, T^\varepsilon] \times \Omega, \\
&x \mapsto u^\varepsilon(t,x) \quad \Omega\text{-periodic} && \text{in } [0, T^\varepsilon], \\
&u^\varepsilon(0,x) = g^0(x), \quad \partial_t u^\varepsilon(0,x) = g^1(x) && \text{in } \Omega,
\end{aligned} \tag{1.12}$$

where g^0, g^1 are given initial conditions and f is a source. The tensor $a(x,y)$ is assumed to be uniformly elliptic and bounded, i.e. there exists $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leq a(x,y)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \ \text{ for a.e. } (x,y) \in \Omega \times Y. \tag{1.13}$$

The well-posedness of (1.12) is proved in [37, 30]. If $g^0 \in \mathrm{W}_{\mathrm{per}}(\Omega)$, $g^1 \in \mathrm{L}_0^2(\Omega)$, $f \in \mathrm{L}^2(0, T^\varepsilon; \mathrm{L}_0^2(\Omega))$, then there exists a unique weak solution $u^\varepsilon \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{W}_{\mathrm{per}}(\Omega))$ with $\partial_t u^\varepsilon \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{L}_0^2(\Omega))$ and $\partial_t^2 u^\varepsilon \in \mathrm{L}^2(0, T^\varepsilon; \mathrm{W}_{\mathrm{per}}^*(\Omega))$.

2. Main result: definition of the family of effective equations and a priori error estimate. In this section we present the main result of the paper. We define the family of effective equations and state the a priori error estimate.

Let us first define the operators involved in the definition of the family of effective equations. For all $x \in \Omega$, let $\{\chi_i(x)\}_{i=1}^d$, $\{\theta_{ij}^0(x)\}_{ij=1}^d$, $\{\theta_i^1(x)\}_{i=1}^d \subset \mathrm{W}_{\mathrm{per}}(Y)$ be the zero mean solutions of the cell problems

$$\big(a(x)\nabla_y \chi_i(x), \nabla_y w\big)_Y = -\big(a(x) e_i, \nabla_y w\big)_Y, \tag{2.1a}$$

$$\big(a(x)\nabla_y \theta_{ij}^0(x), \nabla_y w\big)_Y = -\big(a(x) e_i \chi_j(x), \nabla_y w\big)_Y + \big(a(x)(\nabla_y \chi_j(x) + e_j) - a^0(x) e_j, e_i w\big)_Y, \tag{2.1b}$$

$$\big(a(x)\nabla_y \theta_i^1(x), \nabla_y w\big)_Y = -\big(a(x)\nabla_x \chi_i(x), \nabla_y w\big)_Y + \big(\nabla_x \cdot a(x)(\nabla_y \chi_i(x) + e_i) - \nabla_x \cdot a^0(x) e_i, w\big)_Y, \tag{2.1c}$$

for all test functions $w \in \mathrm{W}_{\mathrm{per}}(Y)$, where $a^0(x)$ is the homogenized tensor defined by

$$a_{ij}^0(x) = \big\langle e_i^T a(x)(\nabla_y \chi_j(x) + e_j)\big\rangle_Y. \tag{2.2}$$

We define the differential operator

$$L^1 = -\partial_i\big(\bar{a}_{ij}^{12}(x)\partial_j\, \cdot\,\big) + b^{10}\partial_t^2, \tag{2.3}$$

based on the following tensors

$$\begin{aligned}
&p_{ijk}^{13}(x) = \big\langle a(x)(\nabla_y \chi_k(x) + e_k) \cdot e_j \chi_i(x)\big\rangle_Y, \\
&q_{ij}^{12}(x) = \big\langle a(x)(\nabla_y \chi_j(x) + e_j) \cdot \nabla_x \chi_i(x)\big\rangle_Y, \\
&\check{a}_{ij}^{12}(x) = S_{ij}^2\Big\{ -\partial_m p_{mij}^{13}(x) + \partial_m p_{jim}^{13}(x) - \partial_m p_{imj}^{13}(x) + 2 q_{ij}^{12}(x)\Big\}, \\
&b^{10} = \max_{x \in \Omega}\left\{ -\frac{\lambda_{\min}(\check{a}^{12}(x))}{\lambda_{\min}(a^0(x))}\right\}_+, \\
&\bar{a}_{ij}^{12}(x) = \check{a}_{ij}^{12}(x) + b^{10} a_{ij}^0(x),
\end{aligned} \tag{2.4}$$

where we denoted $\{\cdot\}_+ = \max\{0, \cdot\}$. Furthermore, we define the differential operator

$$L^2 = \partial_{ij}^2\big(\bar{a}_{ijkl}^{24}(x)\partial_{kl}^2\, \cdot\,\big) - \partial_i\big(b_{ij}^{22}(x)\partial_j\partial_t^2\, \cdot\,\big) - \partial_i\big(\bar{a}_{ij}^{22}(x)\partial_j\, \cdot\,\big) + b^{20}\partial_t^2, \tag{2.5}$$

defined upon the following tensors and functions

$$
\begin{aligned}
&\check{a}^{24}_{ijkl}(x) = S^{2,2}_{ij,kl}\Big\{\big\langle a(x)\chi_i(x)e_j\cdot\chi_l(x)e_k\big\rangle_Y - \big\langle a(x)\nabla_y\theta^0_{ij}(x)\cdot\nabla_y\theta^0_{kl}(x)\big\rangle_Y\Big\},\\
&A^{24}(x) = M\Big(\check{a}^{24}(x)\Big),\quad A^0(x) = M\Big(S^{2,2}_{ij,kl}\big\{a^0_{jk}(x)a^0_{il}(x)\big\}\Big),\\
&\delta \geq \delta^* = \max_{x\in\Omega}\left\{-\frac{\lambda_{\min}(A^{24}(x))}{\lambda_{\min}(A^0(x))}\right\}_+,\\
&\bar{a}^{24}_{ijkl}(x) = \check{a}^{24}_{ijkl}(x) + \delta S^{2,2}_{ij,kl}\big\{a^0_{jk}(x)a^0_{il}(x)\big\},\\
&b^{22}_{ij}(x) = \big\langle\chi_i(x)\chi_j(x)\big\rangle_Y + \delta a^0_{ij}(x),
\end{aligned}
\tag{2.6}
$$

where $S^{2,2}_{ij,kl}\{\cdot\} = S^2_{ij}\{S^2_{kl}\{\cdot\}\}$ and $M(\cdot)$ is given in (1.10) and

$$
\begin{aligned}
&p^{23}_{ijk}(x) = \big\langle a(x)e_j\chi_i(x)\cdot\nabla_x\chi_k(x)\big\rangle_Y - \big\langle a(x)\nabla_y\theta^0_{ji}(x)\cdot\nabla_y\theta^1_k(x)\big\rangle_Y,\\
&p^{22}_{ij}(x) = \big\langle a(x)\nabla_x\chi_j(x)\cdot\nabla_x\chi_i(x)\big\rangle_Y - \big\langle a(x)\nabla_y\theta^1_i(x)\cdot\nabla_y\theta^1_j(x)\big\rangle_Y,\\
&\check{a}^{22}_{ij}(x) = S^2_{ij}\Big\{-\partial_m p^{23}_{mij}(x) + \partial_m p^{23}_{jim}(x) - \partial_m p^{23}_{imj}(x) + p^{22}_{ij}(x)\Big\}\\
&\qquad\qquad + b^{10}\check{a}^{12}_{ij}(x) + \delta\partial_n a^0_{mi}(x)\partial_m a^0_{nj}(x) - \delta\partial_m\big(a^0_{mn}(x)\partial_n a^0_{ij}(x)\big),
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&b^{20} = \max_{x\in\Omega}\left\{-\frac{\lambda_{\min}(\check{a}^{22}(x))}{\lambda_{\min}(a^0(x))}\right\}_+,\\
&\bar{a}^{22}_{ij}(x) = \check{a}^{22}_{ij}(x) + b^{20}a^0_{ij}(x).
\end{aligned}
\tag{2.8}
$$

Observe that the tensors of L^2 are parametrized by $\delta\geq\delta^*$. Let then $\tilde{u}:[0,T^\varepsilon]\times\Omega\to\mathbb{R}$ be the solution of

$$
\begin{aligned}
&\partial_t^2\tilde{u}(t,x) - \partial_i\big(a^0_{ij}(x)\partial_j\tilde{u}(t,x)\big) + \varepsilon L^1\tilde{u}(t,x) + \varepsilon^2L^2\tilde{u}(t,x) = f(t,x) && \text{in } (0,T^\varepsilon]\times\Omega,\\
&x\mapsto\tilde{u}(t,x)\ \ \Omega\text{-periodic} && \text{in } [0,T^\varepsilon],\\
&\tilde{u}(0,x) = g^0(x),\quad \partial_t\tilde{u}(0,x) = g^1(x) && \text{in } \Omega,
\end{aligned}
\tag{2.9}
$$

where the initial conditions g^0, g^1 and the source f are the same as in the equation for u^ε (1.12). It is known that the homogenized tensor a^0 is symmetric, uniformly elliptic, and bounded. Furthermore, note that by definition, $\bar{a}^{12}$, $\bar{a}^{22}, b^{22}$ are symmetric and positive semidefinite, b^{10}, b^{20} are non-negative, and $\bar{a}^{24}$ is major symmetric (i.e. $\bar{a}^{24}_{ijkl} = \bar{a}^{24}_{klij}$) and positive semidefinite (see [8, Lemma 4.2] for a similar result). We verify that if the tensor $a(x,y)$ satisfies $a\in\mathcal{C}^2(\bar{\Omega};\mathrm{L}^\infty(Y))$, then $a^0, \bar{a}^{24}, b^{22}\in\mathcal{C}^2(\bar{\Omega})$, $\bar{a}^{12}\in\mathcal{C}^1(\bar{\Omega})$ and $\bar{a}^{22}\in\mathcal{C}^0(\bar{\Omega})$ (see (4.2)). If, in addition, the data satisfy the regularity

$$
g^0\in\mathrm{W}_{\mathrm{per}}(\Omega)\cap\mathrm{H}^2(\Omega),\quad g^1\in\mathrm{L}^2_0(\Omega)\cap\mathrm{H}^1(\Omega),\quad f\in\mathrm{L}^2(0,T^\varepsilon;\mathrm{L}^2_0(\Omega)),
$$

then there exists a unique weak solution of (2.9) (see e.g. [41, Chapter 2]).

Definition 2.1. *We define the family of effective equations $\mathcal{E}$ as the set of equations* (2.9)*, where a^0 is the homogenized tensor defined in* (2.2) *and L^1, L^2 are defined in* (2.3) *and* (2.5) *for some parameter $\delta\geq\delta^*$.*

Remark 2.2. For uniformly periodic tensors, i.e. $a(x,y) = a(y)\ \forall x\in\Omega$, the family $\mathcal{E}$ simplifies to a parametrized subset of the family defined in [8]. Indeed, in that case we verify that $L^1 = 0$ and $L^2 = \bar{a}^{24}_{ijkl}\partial^4_{ijkl} - b^{22}\partial^2_{ij}\partial^2_t$, where $\bar{a}^{24}, b^{22}$ are constant and satisfy the constraint characterizing the family from [8].

Our main result is the following theorem.

Theorem 2.3. *Assume that the tensor $a(x,y)$ satisfies*

$$
a\in\mathcal{C}^1(\bar{\Omega};\mathrm{W}^{2,\infty}(Y))\cap\mathcal{C}^2(\bar{\Omega};\mathrm{W}^{1,\infty}(Y))\cap\mathcal{C}^4(\bar{\Omega};\mathrm{L}^\infty(Y)).
$$

Furthermore, assume that the solution $\tilde{u}$ of (2.9)*, the initial conditions, and the right hand side satisfy the regularity*

$$
\begin{gathered}
\tilde{u}\in\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^5(\Omega)),\quad \partial_t\tilde{u}\in\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^4(\Omega)),\quad \partial_t^2\tilde{u}\in\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega)),\\
g^0\in\mathrm{H}^4(\Omega),\quad g^1\in\mathrm{H}^4(\Omega),\quad f\in\mathrm{L}^2(0,T^\varepsilon;\mathrm{H}^2(\Omega)).
\end{gathered}
$$

Then the following estimate holds

$$\begin{aligned}\|u^\varepsilon - \tilde u\|_{\mathrm{L}^\infty(0,T^\varepsilon;W)} \le C\varepsilon\Big(&\|g^1\|_{\mathrm{H}^4(\Omega)} + \|g^0\|_{\mathrm{H}^4(\Omega)} + \|f\|_{\mathrm{L}^1(0,T^\varepsilon;\mathrm{H}^2(\Omega))} \\ &+ \textstyle\sum_{k=1}^5 |\tilde u|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^k(\Omega))} + \|\partial_t^2 \tilde u\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))}\Big),\end{aligned} \tag{2.10}$$

where $C = \tilde C\big(\|a\|_{\mathcal{C}^1(\bar\Omega;\mathrm{W}^{2,\infty}Y))} + \|a\|_{\mathcal{C}^2(\bar\Omega;\mathrm{W}^{1,\infty}(Y))} + \|a\|_{\mathcal{C}^4(\bar\Omega;\mathrm{L}^\infty(Y))}\big)$ *and* $\tilde C$ *depends only on* T, λ, Y, *and* δ, *(the norm* $\|\cdot\|_W$ *is defined in* (1.8)*).*

We emphasize that the constant $\tilde C$ is independent of the domain Ω. Hence, if the different norms of the data involved in the estimate are of order $\mathcal{O}(1)$, (2.10) ensures that $\|u^\varepsilon - \tilde u\|_{\mathrm{L}^\infty(0,T^\varepsilon;W)} = \mathcal{O}(\varepsilon)$. In particular, if the data have a spatial support of order $\mathcal{O}(1)$, a reasonable spatial variation, and if f has a temporal support of order $\mathcal{O}(1)$, then $\tilde u$ describes well u^ε over the long time interval $[0, T^\varepsilon]$.

3. Derivations of the adaptation operator and effective equations. In this section, we proceed with the asymptotic expansion and construct the adaptation operator required in the proof of Theorem 2.3, performed in the next section. As we will see, this construction is connected to the operators involved in the family of effective equations defined in Definition 2.1.

The main result of this section is the following theorem.

THEOREM 3.1. *Let* L^1 *and* L^2 *be defined in* (2.3) *and* (2.5)*, respectively. Then there exists an adaptation of the form*

$$\mathcal{B}^\varepsilon \tilde u(t,x) = \tilde u(t,x) + \varepsilon u^1\big(t,x,\tfrac{x}{\varepsilon}\big) + \varepsilon^2 u^2\big(t,x,\tfrac{x}{\varepsilon}\big) + \varepsilon^3 u^3\big(t,x,\tfrac{x}{\varepsilon}\big) + \varepsilon^4 u^4\big(t,x,\tfrac{x}{\varepsilon}\big) + \varphi(t,x), \tag{3.1}$$

such that $x \mapsto \mathcal{B}^\varepsilon \tilde u(t,x)$ *is* Ω*-periodic and*

$$(u^\varepsilon - \mathcal{B}^\varepsilon \tilde u)(0) = \mathcal{O}(\varepsilon), \qquad \partial_t(u^\varepsilon - \mathcal{B}^\varepsilon \tilde u)(0) = \mathcal{O}(\varepsilon), \tag{3.2a}$$

$$(\partial_t^2 + \mathcal{A}^\varepsilon)(u^\varepsilon - \mathcal{B}^\varepsilon \tilde u)(t) = \mathcal{O}(\varepsilon^3) \quad \textit{for a.e. } t \in [0, T^\varepsilon], \tag{3.2b}$$

where we denoted $\mathcal{A}^\varepsilon = -\nabla_x \cdot \big(a\big(x,\tfrac{x}{\varepsilon}\big)\nabla_x \cdot\big)$.

Thanks to (3.2), under sufficient regularity assumptions, the adaptation can be proved to satisfy $\|\,[u^\varepsilon - \mathcal{B}^\varepsilon \tilde u]\,\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathcal{W})} = \mathcal{O}(\varepsilon)$ and Theorem 2.3 is obtained with the triangle inequality (4.1) (this is done rigorously in the next section). Let us note that the accuracy required on the adaptation in (3.2b) is dictated by the order of the timescale $T^\varepsilon = \mathcal{O}(\varepsilon^{-2})$ (see [8] or [41, Chapter 4]).

In the rest of the section, we proceed with the construction of the adaptation $\mathcal{B}^\varepsilon \tilde u$ and of the effective equations. In particular, we need to define the functions u^k and φ in (3.1) so that (3.2) holds. We will see that the definitions of L^1 and L^2 enable the definitions of u^3 to u^4, respectively.

Before entering into technical details, let us present a plan of the construction. First, we formulate the ansatz that an effective equation has the form (2.9), where $a^0(x)$ is the homogeneous tensor (defined in (2.2)) and L^1, L^2 are ε-independent differential operators to be defined. To emphasize that L^1, L^2 are unknown at this point, let us denote them as $\tilde L^1$ and $\tilde L^2$. We then expand $R^\varepsilon = (\partial_t^2 + \mathcal{A}^\varepsilon)(u^\varepsilon - \mathcal{B}^\varepsilon \tilde u)(t)$ with the aim to attain the accuracy (3.2b). Canceling one after another the terms of R^ε of order $\mathcal{O}(\varepsilon^{-1})$ to $\mathcal{O}(\varepsilon^2)$, each u^k takes the form

$$u^k(t,x,y) = \sum_{\ell=1}^k c^{k,\ell}_{i_1\cdot\cdot i_{k-\ell+1}}\big(x,\tfrac{x}{\varepsilon}\big)\partial^{k-\ell+1}_{i_1\cdot\cdot i_{k-\ell+1}}\tilde u(t,x),$$

where the corrector $c^{k,\ell}_{i_1\cdot\cdot i_{k-\ell+1}}(x,\cdot)$ solves a cell problem in Y (i.e., an elliptic PDE with periodic boundary conditions). The well-posedness of these cell problems imposes quantitative constraints on $\tilde L^1$ and $\tilde L^2$. We then design $\tilde L^1$ and $\tilde L^2$ so that these constraints are satisfied and (2.9) is well-posed. In what follows, we require the correctors $c^{k,\ell}_{i_1\cdot\cdot i_{k-\ell+1}}(x,\cdot)$ to have zero mean. While this is a priori not necessary, it is a natural choice and simplifies the computations.

Let us now present the technical details of the derivation. We introduce the differential operators

$$\mathcal{A}_{yy} = -\nabla_y \cdot \big(a(x,y)\nabla_y \cdot\big), \quad \mathcal{A}_{xy} = -\nabla_y \cdot \big(a(x,y)\nabla_x \cdot\big) - \nabla_x \cdot \big(a(x,y)\nabla_y \cdot\big), \quad \mathcal{A}_{xx} = -\nabla_x \cdot \big(a(x,y)\nabla_x \cdot\big).$$

For a sufficiently regular function $\psi(x,y)$, we verify that $\mathcal{A}^\varepsilon\psi\big(x,\frac{x}{\varepsilon}\big) = \big(\varepsilon^{-2}\mathcal{A}_{yy}+\varepsilon^{-1}\mathcal{A}_{xy}+\mathcal{A}_{xx}\big)\psi\big(x,\frac{x}{\varepsilon}\big)$. Hence, using (1.12), (2.9) and (3.1), we obtain the development

$$
\begin{aligned}
R^\varepsilon = (\partial_t^2+\mathcal{A}^\varepsilon)(\mathcal{B}^\varepsilon\tilde{u}-u^\varepsilon)(t,x) = \partial_t^2\mathcal{B}^\varepsilon\tilde{u}(t,x)+\mathcal{A}^\varepsilon\mathcal{B}^\varepsilon\tilde{u}(t,x)-f(t,x)&\\
= \quad \varepsilon^{-1}\big(\qquad \mathcal{A}_{yy}u^1+\mathcal{A}_{xy}\tilde{u} \qquad\qquad\qquad\qquad &\big)\\
+\varepsilon^0 \quad \big(\qquad \mathcal{A}_{yy}u^2+\mathcal{A}_{xy}u^1+\mathcal{A}_{xx}\tilde{u}\;+\partial_i(a^0_{ij}\partial_j\tilde{u}) &\big)\\
+\varepsilon^1 \;\; \big(\partial_t^2u^1+\mathcal{A}_{yy}u^3+\mathcal{A}_{xy}u^2+\mathcal{A}_{xx}u^1-\tilde{L}^1\tilde{u} \qquad &\big)\\
+\varepsilon^2 \;\; \big(\partial_t^2u^2+\mathcal{A}_{yy}u^4+\mathcal{A}_{xy}u^3+\mathcal{A}_{xx}u^2-\tilde{L}^2\tilde{u} \qquad &\big)\\
+(\partial_t^2+\mathcal{A}^\varepsilon)\varphi+\mathcal{O}(\varepsilon^3),&
\end{aligned}
\tag{3.3}
$$

where the u^i are evaluated at $\big(t,x,y=\frac{x}{\varepsilon}\big)$. We then look for $u^1,\dots,u^4$ and φ such that the terms of order $\mathcal{O}(\varepsilon^{-1})$ to $\mathcal{O}(\varepsilon^2)$ in (3.3) vanish. Note that the u^k are set to cancel the terms containing $\tilde{u}$ and φ are set to cancel the terms containing f (that will appear).

3.1. Canceling the $\varepsilon^{-1},\varepsilon^0$ and ε terms and derivation of the constraints defining $\tilde{L}^1$. To cancel the term of order $\mathcal{O}(\varepsilon^{-1})$ in (3.3), it is sufficient to define

$$
u^1(t,x,y) = \chi_i(x,y)\partial_i\tilde{u}(t,x), \tag{3.4}
$$

where, for all $x\in\Omega$ and $1\le i\le d$, $\chi_i(x)\in \mathrm{W}_{\mathrm{per}}(Y)$ solves the cell problem

$$
\varepsilon^{-1}: \quad \big(a(x)\nabla_y\chi_i(x),\nabla_y w\big)_Y = -\big(a(x)e_i,\nabla_y w\big)_Y, \tag{3.5}
$$

for all test functions $w\in\mathrm{W}_{\mathrm{per}}(Y)$. To prove the well-posedness of (3.5), we apply Lax–Milgram theorem. In particular, we must verify that the right hand side belongs to $\mathrm{W}^*_{\mathrm{per}}(\Omega)$. To do so, we need the following characterization (consequence of Riesz representation theorem): a functional $F\in[\mathrm{H}^1_{\mathrm{per}}(Y)]^*$, given by $\langle F,w\rangle = \big(f^0,w\big)_Y+\big(f^1_k,\partial_k w\big)_Y$, for some $f^0,f^1_1,\dots,f^1_d\in\mathrm{L}^2(Y)$, belongs to $\mathrm{W}^*_{\mathrm{per}}(Y)$ if and only if f^0 is zero mean, or equivalently

$$
\big(f^0,1\big)_Y = 0. \tag{3.6}
$$

Using the characterization (3.6), we verify that the right hand side of (3.5) belongs to $\mathrm{W}^*_{\mathrm{per}}(Y)$ and the equation is thus well-posed in $\mathrm{W}_{\mathrm{per}}(Y)$. The equation obtained by canceling the term of order $\mathcal{O}(1)$ in (3.3) reads now

$$
\begin{aligned}
-\nabla_y\cdot(a\nabla_y u^2) = \;& \big(\nabla_y\cdot(e_i\chi_j)+e_i^Ta(\nabla_y\chi_j+e_j)-a^0_{ij}\big)\partial^2_{ij}\tilde{u}\\
&+\big(\nabla_y\cdot(\nabla_x\chi_i)+\nabla_x\cdot a(\nabla_y\chi_i+e_i)-\nabla_x\cdot(a^0e_i)\big)\partial_i\tilde{u}.
\end{aligned}
$$

Compared to the uniformly periodic case in [8], we observe that a supplementary term coming from the variation $x\mapsto a(x,y)$ appears in this equation. To satisfy this equality, it is sufficient to define

$$
u^2(t,x,y) = \theta^0_{ij}(x,y)\partial^2_{ij}\tilde{u}(t,x)+\theta^1_i(x,y)\partial_i\tilde{u}(t,x), \tag{3.7}
$$

where, for all $x\in\Omega$ and $1\le i,j\le d$, $\theta^0_{ij}(x),\theta^1_i(x)\in\mathrm{W}_{\mathrm{per}}(Y)$ satisfy

$$
\varepsilon^0:
$$

$$
\big(a(x)\nabla_y\theta^0_{ij}(x),\nabla_y w\big)_Y = -\big(a(x)e_i\chi_j(x),\nabla_y w\big)_Y+\big(a(x)(\nabla_y\chi_j(x)+e_j)-a^0(x)e_j,e_iw\big)_Y, \tag{3.8a}
$$

$$
\big(a(x)\nabla_y\theta^1_i(x),\nabla_y w\big)_Y = -\big(a(x)\nabla_x\chi_i(x),\nabla_y w\big)_Y+\big(\nabla_x\cdot a(x)(\nabla_y\chi_i(x)+e_i)-\nabla_x\cdot(a^0(x)e_i),w\big)_Y, \tag{3.8b}
$$

for all test functions $w\in\mathrm{W}_{\mathrm{per}}(Y)$. To verify that the right hand sides of these PDEs belong to $\mathrm{W}^*_{\mathrm{per}}(Y)$, we check that they satisfy (3.6). Thanks to the definition of the homogenized tensor a^0 in (2.2), the right hand side of (3.8a) satisfies (3.6) for all $x\in\Omega$. Furthermore, we have

$$
\big(\nabla_x\cdot a(\nabla_y\chi_i+e_i)-\nabla_x\cdot(a^0e_i),1\big)_Y = |Y|\partial_m\Big(\big\langle e_m^Ta(\nabla_y\chi_i+e_i)\big\rangle_Y-a^0_{mi}\Big) = 0,
$$

and the right hand side of (3.8b) also satisfies (3.6). Hence, both cell problems in (3.8) are well-posed. At this point, we have defined an adaptation such that $R^\varepsilon = \mathcal{O}(\varepsilon)$, which would be sufficient for a timescale $\mathcal{O}(1)$. As the timescale is of order $\mathcal{O}(\varepsilon^{-2})$, we need the accuracy $R^\varepsilon = \mathcal{O}(\varepsilon^3)$ in (3.2b), and we thus continue to cancel the higher order terms in (3.3). We begin with the terms containing $\tilde{u}$. Taking into account the definitions of u^1 and u^2 and the effective equation (2.9), we have

$$\begin{aligned}
\partial_t^2 u^1 &= \chi_i \partial_i \partial_t^2 \tilde{u} = \chi_i \partial_i f + \chi_i \partial_{im}(a^0_{mn}\partial_n \tilde{u}) - \varepsilon \chi_i \partial_i \tilde{L}^1 \tilde{u} + \mathcal{O}(\varepsilon^2),\\
\partial_t^2 u^2 &= \theta^0_{ij}\partial^2_{ij}\partial_t^2 \tilde{u} + \theta^1_i \partial_i \partial_t^2 \tilde{u} = \theta^0_{ij}\partial^2_{ij} f + \theta^1_i \partial_i f + \theta^0_{ij}\partial^3_{ijm}(a^0_{mn}\partial_n \tilde{u}) + \theta^1_k \partial_{km}(a^0_{mn}\partial_n\tilde{u}) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Plugging these equalities in (3.3), we obtain

$$\begin{aligned}
R^\varepsilon = \;& \varepsilon\,\big(\mathcal{A}_{yy}u^3 + \mathcal{A}_{xy}u^2 + \mathcal{A}_{xx}u^1 + \chi_i \partial^2_{im}(a^0_{mn}\partial_n\tilde{u}) - \tilde{L}^1\tilde{u}\big)\\
&+ \varepsilon^2\big(\mathcal{A}_{yy}u^4 + \mathcal{A}_{xy}u^3 + \mathcal{A}_{xx}u^2 + \theta^0_{ij}\partial^3_{ijm}(a^0_{mn}\partial_n\tilde{u}) + \theta^1_i\partial^2_{im}(a^0_{mn}\partial_n\tilde{u}) - \chi_i\partial_i\tilde{L}^1\tilde{u} - \tilde{L}^2\tilde{u}\big)\\
&+ (\partial_t^2 + \mathcal{A}^\varepsilon)\varphi + \varepsilon\chi_i\partial_i f + \varepsilon^2(\theta^0_{ij}\partial^2_{ij} f + \theta^1_i \partial_i f) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{3.9}$$

We are now looking for u^3 such that the $\mathcal{O}(\varepsilon)$ order term in (3.9) cancels. We thus define

$$u^3(t,x,y) = \kappa^0_{ijk}(x,y)\partial^3_{ijk}\tilde{u}(t,x) + \kappa^1_{ij}(x,y)\partial^2_{ij}\tilde{u}(t,x) + \kappa^2_i(x,y)\partial_i\tilde{u}(t,x), \tag{3.10}$$

where $\kappa^0_{ijk}(x), \kappa^1_{ij}(x)$ and $\kappa^2_i(x)$ are solutions of cell problems to be defined. We now need to design $\tilde{L}^1$ such that these cell problems are well-posed. The first idea is to set $\tilde{L}^1 = a^{13}_{ijk}(x)\partial^3_{ijk} - a^{12}_{ij}(x)\partial^2_{ij} + a^{11}_i(x)\partial_i$ and to define the tensors a^{13}, a^{12}, a^{11} using the constraints imposed by the solvability of the cell problems. However, we also have to ensure the well-posedness of the effective equation (2.9). We will see that $a^{13}_{ijk}(x)\partial^3_{ijk} = 0$. Nevertheless, for the operator $-\varepsilon a^{12}_{ij}(x)\partial^2_{ij}$ not to deteriorate the ellipticity of $-\partial_i(a^0_{ij}\partial_j\cdot)$, a^{12} has to be non-negative. This condition can not be ensured in general by the tensor involved by the obtained constraint. We thus apply a Boussinesq trick: adding the term $b^{10}\partial_t^2$ in $\tilde{L}^1$, we observe that if we formally substitute $\partial_t^2\tilde{u} = f + \partial_i(a^0_{ij}\partial_j\tilde{u})$ in $\tilde{L}^1\tilde{u}$, the constraint imposed by the well-posedness of the cell problem for κ^1_{ij} applies on $a^{12}_{ij} - b^{10}a^0_{ij}$. As a^0 is positive definite, note that we can then find non-negative b^{10}, a^{12} that satisfy the constraint. Let then

$$\tilde{L}^1 = a^{13}_{ijk}(x)\partial^3_{ijk} - a^{12}_{ij}(x)\partial^2_{ij} + a^{11}_i(x)\partial_i + b^{10}(x)\partial_t^2. \tag{3.11}$$

Using the effective equation, we obtain

$$\tilde{L}^1\tilde{u} = \tilde{L}^{1,x}\tilde{u} + b^{10}\partial_m(a^0_{mn}\partial_n\tilde{u}) + b^{10}f + \varepsilon b^{10}\tilde{L}^1\tilde{u} + \mathcal{O}(\varepsilon^2), \tag{3.12}$$

where we denoted $\tilde{L}^{1,x} = \tilde{L}^1 - b^{10}(x)\partial_t^2$, the spatial part of $\tilde{L}^1$. Hence, we rewrite (3.9) as

$$\begin{aligned}
R^\varepsilon = \;& \varepsilon\,\big(\mathcal{A}_{yy}u^3 + \mathcal{A}_{xy}u^2 + \mathcal{A}_{xx}u^1 + \chi_i\partial^2_{im}(a^0_{mn}\partial_n\tilde{u}) - \tilde{L}^{1,x}\tilde{u} - b^{10}\partial_m(a^0_{mn}\partial_n\tilde{u})\big)\\
&+ \varepsilon^2\big(\mathcal{A}_{yy}u^4 + \mathcal{A}_{xy}u^3 + \mathcal{A}_{xx}u^2 + \theta^0_{ij}\partial^3_{ijm}(a^0_{mn}\partial_n\tilde{u}) + \theta^1_i\partial^2_{im}(a^0_{mn}\partial_n\tilde{u})\\
&\qquad - \chi_i\partial_i(\tilde{L}^1\tilde{u}) + b^{10}\tilde{L}^1\tilde{u} - \tilde{L}^2\tilde{u}\big)\\
&+ (\partial_t^2 + \mathcal{A}^\varepsilon)\varphi + \varepsilon\chi_i\partial_i f + \varepsilon^2(\theta^0_{ij}\partial^2_{ij}f + \theta^1_i\partial_i f) - \varepsilon b^{10} f + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{3.13}$$

Recalling the definition of u^3 in (3.10), the cancellation of the $\mathcal{O}(\varepsilon)$ order term in (3.13) leads to the following cell problems: for all $x \in \Omega$ and $1 \le i,j,k \le d$, $\kappa^0_{ijk}(x), \kappa^1_{ij}(x), \kappa^2_i(x) \in \mathrm{W}_{\mathrm{per}}(Y)$ satisfy (for readability we do not specify the evaluation in x)

ε^1 :

$$\big(a\nabla_y\kappa^0_{ijk}, \nabla_y w\big)_Y = -\big(ae_i\theta^0_{jk}, \nabla_y w\big)_Y + \big(a(\nabla_y\theta^0_{jk} + e_j\chi_k), e_i w\big)_Y - \big(a^0_{ij}\chi_k, w\big)_Y + \big(a^{13}_{ijk}, w\big)_Y, \tag{3.14a}$$

$$\begin{aligned}
\big(a\nabla_y\kappa^1_{ij}, \nabla_y w\big)_Y = &-\big(a(\nabla_x\theta^0_{ij} + e_i\theta^1_j), \nabla_y w\big)_Y + \big(\nabla_x\cdot a(\nabla_y\theta^0_{ij} + e_i\chi_j), w\big)_Y\\
&+ \big(a(\nabla_y\theta^1_j + \nabla_x\chi_j), e_i w\big)_Y - \big(\chi_i\partial_m a^0_{mj} + \chi_m\partial_m a^0_{ij}, w\big)_Y - \big(a^{12}_{ij} - b^{10}a^0_{ij}, w\big)_Y,
\end{aligned} \tag{3.14b}$$

$$\begin{aligned}
\big(a\nabla_y\kappa^2_i, \nabla_y w\big)_Y = &-\big(a\nabla_x\theta^1_i, \nabla_y w\big)_Y + \big(\nabla_x\cdot a(\nabla_y\theta^1_i + \nabla_x\chi_i), w\big)_Y - \big(\chi_m\partial^2_{mn}a^0_{ni}, w\big)_Y\\
&+ \big(b^{10}\partial_m a^0_{mi} + a^{11}_i, w\big)_Y,
\end{aligned} \tag{3.14c}$$

for all test functions $w \in \mathrm{W}_{\mathrm{per}}(Y)$. These cell problems are well-posed if their right hand sides satisfy (3.6). This is the case if and only if the tensors of $\tilde{L}^1$ satisfy the following constraints (recall that $\langle \chi_k(x) \rangle_Y = 0$):

$$|Y| a^{13}_{ijk} = -\big(a(\nabla_y \theta^0_{jk} + e_j \chi_k), e_i\big)_Y, \tag{3.15a}$$

$$|Y|(a^{12}_{ij} - b^{10} a^0_{ij}) = \big(\nabla_x \cdot a(\nabla_y \theta^0_{ij} + e_i \chi_j), 1\big)_Y + \big(a(\nabla_y \theta^1_j + \nabla_x \chi_j), e_i\big)_Y, \tag{3.15b}$$

$$|Y| a^{11}_i = -\big(\nabla_x \cdot a(\theta^1_i + \nabla_x \chi_i), 1\big)_Y - |Y| b^{10} \partial_m a^0_{mi}. \tag{3.15c}$$

We emphasize that the constraints (3.15) must hold for each $x \in \Omega$. These expressions are simplified in the following lemma.

Lemma 3.2. *The constraints on a^{13}, a^{12}, b^{10} and a^{11} defined in* (3.15) *can be rewritten for all $x \in \Omega$ as*

$$a^{13}_{ijk}(x) = (p^{13}_{ijk} - p^{13}_{kji})(x), \qquad p^{13}_{ijk} = \big\langle a(\nabla_y \chi_k + e_k) \cdot e_j \chi_i \big\rangle_Y, \tag{3.16a}$$

$$(a^{12}_{ij} - b^{10} a^0_{ij})(x) = -\partial_m a^{13}_{mij}(x) + p^{12}_{ij}(x), \qquad p^{12}_{ij} = \big\langle a(\nabla_y \theta^1_j + \nabla_x \chi_j) \cdot e_i \big\rangle_Y, \tag{3.16b}$$

$$a^{11}_i(x) = -\partial_m p^{12}_{mi}(x) - b^{10} \partial_m a^0_{mi}(x). \tag{3.16c}$$

Furthermore, $p^{12}(x)$ satisfies

$$p^{12}_{ij}(x) = -\partial_m p^{13}_{imj}(x) + q^{12}_{ij}(x) + q^{12}_{ji}(x), \qquad q^{12}_{ij} = \big\langle a(\nabla_y \chi_j + e_j) \cdot \nabla_x \chi_i \big\rangle_Y. \tag{3.16d}$$

Proof. Let us denote $(\cdot,\cdot)_Y$ as $(\cdot,\cdot)$ and $\langle \cdot \rangle_Y$ as $\langle \cdot \rangle$. We first prove (3.16a). Using (3.5) with the test function $w = \theta^0_{jk}$ and (3.8a) with $w = \chi_i$, we have

$$-\big(a(\nabla_y \theta^0_{jk} + e_j \chi_k), e_i\big) = \big(a \nabla_y \theta^0_{jk}, \nabla_y \chi_i\big) - \big(a e_j \chi_k, e_i\big) = -\big(a e_j \chi_k, \nabla_y \chi_i + e_i\big) + \big(a(\nabla_y \chi_k + e_k), e_j \chi_i\big),$$

which, thanks to the symmetry of $a(x,y)$ proves (3.16a). Let us now prove (3.16b). Thanks to (3.15a), the first term of (3.15b) is

$$\big(\nabla_x \cdot a(\nabla_y \theta^0_{ij} + e_i \chi_j), 1\big) = \partial_m \big(a(\nabla_y \theta^0_{ij} + e_i \chi_j), e_m\big) = -|Y| \partial_m a^{13}_{mij},$$

and thus (3.15b) can be rewritten as (3.16b). To rewrite a^{11}_i as in (3.16c), we simply note that $-\big(\nabla_x \cdot a(\nabla_y \theta^1_i + \nabla_x \chi_i), 1\big) = -|Y| \partial_m p^{12}_{mi}$. Finally, let us prove (3.16d). Using (3.5) with the test function $w = \theta^1_j$ and (3.8b) with $w = \chi_i$, we have

$$\big(a(\nabla_y \theta^1_j + \nabla_x \chi_j), e_i\big) = -\big(a \nabla_y \theta^1_j, \nabla_y \chi_i\big) + \big(a \nabla_x \chi_j, e_i\big) = \big(a \nabla_x \chi_j, \nabla_y \chi_i + e_i\big) - \big(\nabla_x \cdot a(\nabla_y \chi_j + e_j), \chi_i\big).$$

Furthermore, the last term satisfies

$$-\big(\nabla_x \cdot a(\nabla_y \chi_j + e_j), \chi_i\big) = -\partial_m \big(a(\nabla_y \chi_j + e_j), e_m \chi_i\big) + \big(a(\nabla_y \chi_j + e_j), \nabla_x \chi_i\big) = |Y|(-\partial_m p^{13}_{imj} + q^{12}_{ij}).$$

Combining the two last equalities gives (3.16d) and the proof of the lemma is complete. ∎

In the following proposition, we verify that the two operators $\tilde{L}^1$ and L^1 coincide.

Proposition 3.3. *Let $\bar{a}^{12}$ and b^{10} be the tensors defined in* (2.4) *and assume that $\bar{a}^{12} \in \mathcal{C}^1(\bar{\Omega})$. Let also $\tilde{L}^1$ and L^1 be the operators defined in* (3.11) *and* (2.3)*, respectively. Then $\tilde{L}^1 v = L^1 v$ for any $v \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^3(\Omega))$ with $\partial_t^2 v \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{L}^2(\Omega))$.*

Proof. First, note that thanks to (3.16a), we have $S^3_{ijk}\{a^{13}_{ijk}\} = 0$ and thus $a^{13}_{ijk} \partial^3_{ijk} v = 0$. Furthermore, thanks to (3.16a), (3.16b), and (3.16d), we verify that $S^2_{ij}\{a^{12}_{ij}\} = \bar{a}^{12}_{ij}$. Hence, we have

$$\tilde{L}^1 v - b^{10} \partial_t^2 v = -S^2_{ij}\{a^{12}_{ij}\} \partial^2_{ij} v + a^{11}_i \partial_i v = -\partial_i \big(\bar{a}^{12}_{ij} \partial_j v\big) + \big(a^{11}_i + \partial_m (S^2_{mi}\{a^{12}_{mi}\})\big) \partial_i v. \tag{3.17}$$

We claim that $a^{11}_i + \partial_m (S^2_{mi}\{a^{12}_{mi}\}) = 0$. To prove it, note that as b^{10} is constant, using (3.16b) and (3.16c), we have

$$a^{11}_i + \partial_m (S^2_{mi}\{a^{12}_{mi}\}) = \tfrac{1}{2} \partial_m \big(p^{12}_{im} - p^{12}_{mi}\big) - \tfrac{1}{2} \partial^2_{mn} \big(a^{13}_{nmi} + a^{13}_{nim}\big).$$

Using then (3.16a) and (3.16d), we verify that

$$a^{11}_i + \partial_m (S^2_{mi}\{a^{12}_{mi}\}) = \tfrac{1}{2} \partial^2_{mn} \big(- p^{13}_{inm} + p^{13}_{mni} - p^{13}_{nmi} + p^{13}_{imn} - p^{13}_{nim} + p^{13}_{min}\big) = 0,$$

and the claim is proved. Combined with (3.17), the claim concludes the proof of the lemma. ∎

3.2. Canceling the ε^2 term and derivation of the constraints defining $\tilde{L}^2$. We now come back to the asymptotic expansion. The next step is to cancel the $\mathcal{O}(\varepsilon^2)$ order term containing $\tilde{u}$ in (3.13). Following the same reasoning as for u^3, we define u^4 as

$$u^4(t,x,y) = \rho^0_{ijkl}(x,y)\partial^4_{ijkl}\tilde{u}(t,x) + \rho^1_{ijk}(x,y)\partial^3_{ijk}\tilde{u}(t,x) + \rho^2_{ij}(x,y)\partial^2_{ij}\tilde{u}(t,x) + \rho^3_i(x,y)\partial_i\tilde{u}(t,x), \tag{3.18}$$

for some $\rho^0, \rho^1, \rho^2, \rho^3$ to be defined. The ansatz on the form of $\tilde{L}^2$ could be $\partial^2_{ij}(a^{24}_{ijkl}\partial^2_{kl}) + a^{23}_{ijk}\partial^3_{ijk} - a^{22}_{ij}\partial^2_{ij} + a^{21}_i\partial_i$. However, as for $\tilde{L}^1$, this choice does not allow the well-posedness of the effective equation (2.9). We thus apply Boussinesq tricks. First, similarly as for $\tilde{L}^1$, we add the operator $b^{20}\partial^2_t$ in $\tilde{L}^2$ in order to obtain a constraint on the difference $a^{22}_{ij} - b^{20}a^0_{ij}$. Second, inspired by the uniformly periodic case in [8], we add the term $-\partial_i(b^{22}_{ij}\partial_j\partial^2_t\cdot)$ in order to obtain a constraint on $a^{24}_{ijkl} - a^0_{jk}b^{22}_{il}$. Finally, for the operator of order 3, we will see that we can find a tensor a^{23} that satisfies the constraint and $a^{23}_{ijk}\partial^3_{ijk} = 0$. We thus define

$$\tilde{L}^2 = \partial^2_{ij}(a^{24}_{ijkl}(x)\partial^2_{kl}\cdot) - \partial_i(b^{22}_{ij}(x)\partial_j\partial^2_t\cdot) + a^{23}_{ijk}(x)\partial^3_{ijk} - a^{22}_{ij}(x)\partial^2_{ij} + a^{21}_i(x)\partial_i + b^{20}(x)\partial^2_t, \tag{3.19}$$

and, using (2.9), we verify that

$$\tilde{L}^2\tilde{u} = \tilde{L}^{2,x}\tilde{u} - \partial_i(b^{22}_{ij}\partial^2_{jk}(a^0_{kl}\partial_l\tilde{u})) + b^{20}\partial_m(a^0_{mn}\partial_n\tilde{u}) - \partial_i(b^{22}_{ij}\partial_j f) + b^{20}f + \mathcal{O}(\varepsilon), \tag{3.20}$$

where $\tilde{L}^{2,x} = \tilde{L}^2 + \partial_i(b^{22}_{ij}\partial_j\partial^2_t\cdot) - b^{20}\partial^2_t$ is the spatial part of $\tilde{L}^2$. Let us rewrite the following terms of (3.13) taking into account the definition of $\tilde{L}^1$ and using (2.9):

$$\begin{aligned}
&\chi_i\partial_i(\tilde{L}^1\tilde{u}) = \chi_i\partial_i(\tilde{L}^{1,x}\tilde{u}) + \chi_i\partial_i(b^{10}\partial_m(a^0_{mn}\partial_n\tilde{u})) + \chi_i\partial_i(b^{10}f) + \mathcal{O}(\varepsilon),\\
&b^{10}\tilde{L}^1\tilde{u} = b^{10}\tilde{L}^{1,x}\tilde{u} + (b^{10})^2\partial_m(a^0_{mn}\partial_n\tilde{u}) + (b^{10})^2 f + \mathcal{O}(\varepsilon).
\end{aligned}$$

Therefore, using the definitions of the u^k, and (3.20), we rewrite the $\mathcal{O}(\varepsilon^2)$ order term in (3.13) and obtain

$$\begin{aligned}
R^\varepsilon = \varepsilon^2\big(&\mathcal{A}_{yy}u^4 + \mathcal{A}_{xy}u^3 + \mathcal{A}_{xx}u^2 + \theta^0_{ij}\partial^3_{ijm}(a^0_{mn}\partial_n\tilde{u}) + \theta^1_i\partial^2_{im}(a^0_{mn}\partial_n\tilde{u}) - \chi_i\partial_i(\tilde{L}^{1,x}\tilde{u})\\
&+ \chi_i\partial_i(b^{10}\partial_m(a^0_{mn}\partial_n\tilde{u})) - b^{10}\tilde{L}^{1,x}\tilde{u} + (b^{10})^2\partial_m(a^0_{mn}\partial_n\tilde{u})\\
&- \tilde{L}^{2,x}\tilde{u} + \partial_i(b^{22}_{ij}\partial^2_{jm}(a^0_{mn}\partial_n\tilde{u})) - b^{20}\partial_m(a^0_{mn}\partial_n\tilde{u})\big)\\
&+ \varepsilon(\chi_i\partial_i f - b^{10}f) + \varepsilon^2\big(\theta^0_{ij}\partial^2_{ij}f + \theta^1_i\partial_i f - \chi_i\partial_i(b^{10}f) + (b^{10})^2 f + \partial_i(b^{22}_{ij}\partial_j f) - b^{20}f\big)\\
&+(\partial^2_t + \mathcal{A}^\varepsilon)\varphi + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{3.21}$$

We thus obtain the following cell problems: for $x \in \Omega$ and $1 \le i,j,k,l \le d$, $\rho^0_{ijkl}(x)$, $\rho^1_{ijk}(x)$, $\rho^2_{ij}(x)$, $\rho^3_i(x) \in \mathrm{W}_{\mathrm{per}}(Y)$ satisfy

$$\varepsilon^2:$$

$$\begin{aligned}
\big(a\nabla_y\rho^0_{ijkl}, \nabla_y w\big)_Y = &- \big(ae_i\kappa^0_{jkl}, \nabla_y w\big)_Y + \big(a(\nabla_y\kappa^0_{jkl} + e_j\theta^0_{kl}), e_i w\big)_Y\\
&+ \big(a^{13}_{jkl}\chi_i - a^0_{ij}\theta^0_{kl}, w\big)_Y + \big(a^{24}_{ijkl} - a^0_{jk}b^{22}_{il}, w\big)_Y,
\end{aligned} \tag{3.22a}$$

$$\begin{aligned}
\big(a\nabla_y\rho^1_{ijk}, \nabla_y w\big)_Y = &- \big(a(e_i\kappa^1_{jk} + \nabla_x\kappa^0_{ijk}, \nabla_y w\big)_Y + \big(\nabla_x\cdot a(\nabla_y\kappa^0_{ijk} + e_i\theta^0_{jk}), w\big)_Y\\
&+ \big(a(\nabla_y\kappa^1_{jk} + \nabla_x\theta^0_{jk} + e_j\theta^1_k), e_i w\big)_Y\\
&+ \big(\chi_m\partial_m a^{13}_{ijk} + (b^{10}a^0_{ij} - a^{12}_{ij})\chi_k - \theta^0_{ij}\partial_m a^0_{mk} - \theta^0_{mi}\partial_m a^0_{jk} - \theta^0_{im}\partial_m a^0_{jk} - a^0_{ij}\theta^1_k, w\big)_Y\\
&+ \big(\partial_m(a^{24}_{imjk} + a^{24}_{mijk} - b^{22}_{mk}a^0_{ij}) - b^{22}_{im}\partial_m a^0_{jk} - b^{22}_{ij}\partial_m a^0_{mk} - b^{10}a^{13}_{ijk} + a^{23}_{ijk}, w\big)_Y,
\end{aligned} \tag{3.22b}$$

$$\begin{aligned}
\big(a\nabla_y\rho^2_{ij}, \nabla_y w\big)_Y = &- \big(a(e_i\kappa^2_j + \nabla_x\kappa^1_{ij}), \nabla_y w\big) + \big(\nabla_x\cdot a(\nabla_y\kappa^1_{ij} + \nabla_x\theta^0_{ij} + e_i\theta^1_j), w\big)_Y\\
&+ \big(a(\nabla_y\kappa^2_j + \nabla_x\theta^1_j), e_i w\big)_Y\\
&+ \big(\chi_m\partial_m(b^{10}a^0_{ij} - a^{12}_{ij}) + b^{10}\chi_i\partial_m a^0_{mj} + a^{11}_i\chi_j\\
&\qquad - \theta^1_i\partial_m a^0_{mj} - \theta^1_m\partial_m a^0_{ij} - (\theta^0_{im} + \theta^0_{mi})\partial^2_{mn}a^0_{nj} - \theta^0_{mn}\partial^2_{mn}a^0_{ij}, w\big)_Y\\
&+ \big(\partial^2_{mn}a^{24}_{mnij} - \partial_m(b^{22}_{mi}\partial_n a^0_{nj}) - \partial_m(b^{22}_{mn}\partial_n a^0_{ij}) - b^{22}_{im}\partial^2_{mn}a^0_{nj}\\
&\qquad + b^{10}(a^{12}_{ij} - b^{10}a^0_{ij}) - (a^{22}_{ij} - b^{20}a^0_{ij}), w\big)_Y,
\end{aligned} \tag{3.22c}$$

$$\begin{aligned}\big(a\nabla_y\rho_i^3,\nabla_y w\big)_Y = &-\big(a\nabla_x\kappa_i^2,\nabla_y w\big)_Y + \big(\nabla_x\cdot a(\nabla_y\kappa_i^2+\nabla_x\theta_i^1),w\big)_Y\\ &+\big(\chi_m\partial_m a_i^{11}+\chi_m\partial_m(b^{10}\partial_n a_{ni}^0)-\theta_{mn}^0\partial_{mnp}^3 a_{pi}^0-\theta_m^1\partial_{mn}^2 a_{ni}^0,w\big)_Y\\ &+\big(b^{20}\partial_m a_{mi}^0-b^{10}(b^{10}\partial_m a_{mi}^0+a_i^{11})-\partial_m(b_{mn}^{22}\partial_{np}^2 a_{pi}^0)+a_i^{21},w\big)_Y,\end{aligned}\tag{3.22d}$$

for all test functions $w\in \mathrm{W}_{\mathrm{per}}(Y)$. The cell problems (3.22) are well-posed if their right hand sides satisfy (3.6). This is the case if and only if the tensors of $\tilde{L}^2$ satisfy the following constraints ($\chi_i(x),\theta_{ij}^0(x)$ and $\theta_i^1(x)$ have zero mean):

$$|Y|(a_{ijkl}^{24}-a_{jk}^0 b_{il}^{22}) = -\big(a(\nabla_y\kappa_{jkl}^0+e_j\theta_{kl}^0),e_i\big)_Y,\tag{3.23a}$$

$$\begin{aligned}|Y|a_{ijk}^{23} = &-\big(\nabla_x\cdot a(\nabla_y\kappa_{ijk}^0+e_i\theta_{jk}^0),1\big)_Y-\big(a(\nabla_y\kappa_{jk}^1+\nabla_x\theta_{jk}^0+e_j\theta_k^1),e_i\big)_Y\\ &+|Y|\big(b_{im}^{22}\partial_m a_{jk}^0+b_{ij}^{22}\partial_m a_{mk}^0+b^{10}a_{ijk}^{13}+\partial_m(b_{mk}^{22}a_{ij}^0-a_{imjk}^{24}-a_{mijk}^{24})\big),\end{aligned}\tag{3.23b}$$

$$\begin{aligned}|Y|(a_{ij}^{22}-b^{20}a_{ij}^0) = &\big(\nabla_x\cdot a(\nabla_y\kappa_{ij}^1+\nabla_x\theta_{ij}^0+e_i\theta_j^1),1\big)_Y+\big(a(\nabla_y\kappa_j^2+\nabla_x\theta_j^1),e_i\big)_Y\\ &+|Y|\big(\partial_{mn}^2 a_{mnij}^{24}-\partial_m(b_{mi}^{22}\partial_n a_{nj}^0)-\partial_m(b_{mn}^{22}\partial_n a_{ij}^0)-b_{im}^{22}\partial_{mn}^2 a_{nj}^0\\ &+b^{10}(a_{ij}^{12}-b^{10}a_{ij}^0)\big),\end{aligned}\tag{3.23c}$$

$$|Y|a_i^{21} = -\big(\nabla_x\cdot a(\nabla_y\kappa_i^2+\nabla_x\theta_i^1),1\big)_Y+|Y|\big(b^{10}(b^{10}\partial_m a_{mi}^0+a_i^{11})-b^{20}\partial_m a_{mi}^0+\partial_m(b_{mn}^{22}\partial_{np}^2 a_{pi}^0)\big),\tag{3.23d}$$

for all $x\in\Omega$. These expressions are simplified in the following Lemma (we refer to [41, Lemma 6.2.8] for the proof).

Lemma 3.4. *Denote $R_{ij}(x)=b_{ij}^{22}(x)-\langle\chi_j(x)\chi_i(x)\rangle_Y$. Then the constraints on a^{24}, b^{22}, a^{23}, a^{22}, b^{20} and a^{21} given in* (3.23) *can be rewritten as*

$$a_{ijkl}^{24} = \langle a_{jk}\chi_l\chi_i\rangle_Y-\langle a\nabla_y\theta_{kl}^0\cdot\nabla_y\theta_{ji}^0\rangle_Y+a_{jk}^0R_{il},\tag{3.24a}$$

$$a_{ijk}^{23} = p_{ijk}^{23}-p_{kji}^{23}+b^{10}a_{ijk}^{13}-\partial_m(a_{mj}^0R_{ik})+\partial_m a_{mk}^0R_{ij}+\partial_m a_{jk}^0R_{mi},\tag{3.24b}$$

$$p_{ijk}^{23} = \langle ae_j\chi_i\cdot\nabla_x\chi_k\rangle_Y-\langle a\nabla_y\theta_{ji}^0\cdot\nabla_y\theta_k^1\rangle_Y,\tag{3.24c}$$

$$\begin{aligned}a_{ij}^{22}-b^{20}a_{ij}^0 = &\partial_m(p_{jim}^{23}-p_{mij}^{23}-p_{imj}^{23})+p_{ij}^{22}+b^{10}(a_{ij}^{12}-b^{10}a_{ij}^0)\\ &+\partial_{mn}^2(a_{ni}^0R_{mj})-\partial_m(\partial_n a_{nj}^0R_{mi})-\partial_m(\partial_n a_{ij}^0R_{mn})-\partial_{mn}^2 a_{nj}^0R_{im},\end{aligned}\tag{3.24d}$$

$$p_{ij}^{22} = \langle a\nabla_x\chi_j\cdot\nabla_x\chi_i\rangle_Y-\langle a\nabla_y\theta_i^1\cdot\nabla_y\theta_j^1\rangle_Y,\tag{3.24e}$$

$$a_i^{21} = \partial_{mn}^2 p_{mni}^{23}-\partial_m p_{mi}^{22}+b^{10}(b^{10}\partial_m a_{mi}^0+a_i^{11})-b^{20}\partial_m a_{mi}^0+\partial_m(\partial_{np}^2 a_{pi}^0R_{mn}).\tag{3.24f}$$

We then verify that the two operators $\tilde{L}^2$ and L^2 coincide.

Proposition 3.5. *Let $\bar{a}^{24}$, b^{22}, $\bar{a}^{22}$ be the tensors defined in* (2.6) *and* (2.8) *and assume that $\bar{a}^{24}\in\mathcal{C}^2(\bar{\Omega})$ and $b^{22},\bar{a}^{12}\in\mathcal{C}^1(\bar{\Omega})$. Let also L^2 be the operator defined in* (2.5) *and $\tilde{L}^2$ be the operator defined in* (3.19) *with the tensors given in* (3.24) *where $R_{ij}=\delta a_{ij}^0$ for some $\delta\in\mathbb{R}$. Then $\tilde{L}^2v=L^2v$ for any $v\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^4(\Omega))$ with $\partial_t^2v\in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^2(\Omega))$.*

Proof. First, inserting $R_{ij}=\delta a_{ij}^0$ in (3.24d) and using (3.16a), we verify that $S_{ijk}^3\{a_{ijk}^{23}\}=0$ and thus $a_{ijk}^{23}\partial_{ijk}^3v=0$. Second, using (3.24d), (2.4), and the definition of R_{ij}, we verify that $S_{ij}^2\{a_{ij}^{22}\}=\bar{a}_{ij}^{22}$. Furthermore, it holds $S_{ij,kl}^{2,2}\{a_{ijkl}^{24}\}=\bar{a}_{ijkl}^{24}$. Hence, denoting $\tilde{L}^{2,x}=\tilde{L}^2+\partial_i\big(b_{ij}^{22}\partial_j\partial_t^2\,\cdot\big)-b^{20}\partial_t^2$, we have

$$\tilde{L}^{2,x}v=\partial_{ij}^2\big(\bar{a}_{ijkl}^{24}\partial_{kl}^2v\big)-\partial_i\big(S_{ij}^2\{a_{ij}^{22}\}\partial_jv\big)+\big(a_i^{21}+\partial_m(S_{mi}^2\{a_{mi}^{22}\})\big)\partial_iv.\tag{3.25}$$

We claim that $a_i^{21}+\partial_m(S_{mi}^2\{a_{mi}^{22}\})=0$. Indeed, using (3.24d), the form of R_{ij}, and the symmetry of p^{22} and a^0, we compute

$$S_{ij}^2\{a_{ij}^{22}\}=S_{ij}^2\big\{\partial_n(p_{jin}^{23}-p_{nij}^{23}-p_{inj}^{23})\big\}+p_{ij}^{22}+b^{10}S_{ij}^2\{a_{ij}^{12}\}-(b^{10})^2a_{ij}^0+\delta\partial_n a_{pi}^0\partial_p a_{nj}^0-\delta\partial_p\big(a_{pn}^0\partial_n a_{ij}^0\big)+b^{20}a_{ij}^0.$$

Note that we have seen in the proof of Proposition 3.3 that $a_i^{11}+\partial_m(S_{mi}^2\{a_{mi}^{12}\})=0$. Using then (3.24f), direct computations lead to

$$a_i^{21}+\partial_m(S_{mi}^2\{a_{mi}^{22}\})=\delta\partial_m\big(\partial_{np}^2a_{pi}^0a_{mn}^0\big)+\delta\partial_m\big(\partial_n a_{pm}^0\partial_p a_{ni}^0\big)-\delta\partial_{mp}^2\big(a_{pn}^0\partial_n a_{mi}^0\big)=0,$$

which proves the claim. Combined with (3.25), the claim concludes the proof of the lemma. □

3.3. Including a non-zero right hand side. To reach the accuracy $R^\varepsilon = \mathcal{O}(\varepsilon^3)$ in (3.21), we still have to remove the terms coming from the right hand side f. To do so, we let φ in (3.1) belongs to the unique class of solution $\boldsymbol{\varphi}$ of the equation

$$\begin{aligned}&(\partial_t^2 + \mathcal{A}^\varepsilon)\boldsymbol{\varphi}(t) = -\boldsymbol{\mathcal{S}}^\varepsilon f(t) \quad \text{in } \mathcal{W}_{\text{per}}^*(\Omega) \quad \text{for a.e.} \quad t \in [0, T^\varepsilon],\\ &\boldsymbol{\varphi}(0) = \partial_t \boldsymbol{\varphi}(0) = \big[0\big],\end{aligned} \tag{3.26}$$

where, denoting $\chi_i^\varepsilon = \chi_i\big(\cdot, \tfrac{\cdot}{\varepsilon}\big), \theta_{ij}^{0\varepsilon} = \theta_{ij}^0\big(\cdot, \tfrac{\cdot}{\varepsilon}\big), \theta_i^{1\varepsilon} = \theta_i^1\big(\cdot, \tfrac{\cdot}{\varepsilon}\big)$,

$$\boldsymbol{\mathcal{S}}^\varepsilon f = \big[\varepsilon(\chi_i^\varepsilon \partial_i f - b^{10} f) + \varepsilon^2(\theta_{ij}^{0\varepsilon}\partial_{ij}^2 f + \theta_i^{1\varepsilon}\partial_i f - \chi_i^\varepsilon \partial_i(b^{10} f) + (b^{10})^2 f + \partial_i(b_{ij}^{22}\partial_j f) - b^{20} f)\big].$$

We verify that $\boldsymbol{\varphi} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathcal{W}_{\text{per}}(\Omega))$, $\partial_t \boldsymbol{\varphi} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathcal{L}^2(\Omega))$ and $\partial_t^2 \boldsymbol{\varphi} \in \mathrm{L}^2(0, T^\varepsilon; \mathcal{W}_{\text{per}}^*(\Omega))$. Furthermore, the standard energy estimate for the wave equation ensures that

$$\|\boldsymbol{\varphi}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathcal{W})} \le \|\nabla_x \boldsymbol{\varphi}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \le C\varepsilon \|f\|_{\mathrm{L}^1(0,T^\varepsilon;\mathrm{H}^2(\Omega))}, \tag{3.27}$$

where C depends only on

$$\lambda, \Lambda, \|\chi_i\|_{\mathcal{C}^0(\bar{\Omega};\mathcal{C}^0(Y))}, \|b_{ij}^{22}\|_{\mathcal{C}^1(\bar{\Omega})}, |b^{10}|, |b^{20}|, \|\theta_{ij}^0\|_{\mathcal{C}^0(\bar{\Omega};\mathcal{C}^0(Y))}, \|\theta_i^1\|_{\mathcal{C}^0(\bar{\Omega};\mathcal{C}^0(Y))}.$$

3.4. Proof of Theorem 3.1. To conclude this section, let us prove Theorem 3.1. The adaptation $\mathcal{B}^\varepsilon\tilde{u}$ is defined by (3.1), where $u^1, \dots, u^4$ are defined in (3.4), (3.7), (3.10), and (3.18), and $\varphi \in \boldsymbol{\varphi}$ solves (3.26). Then, combining Lemma 3.2 with Proposition 3.3, we verify that $L^1\tilde{u} = \tilde{L}^1\tilde{u}$, where the tensors involved in the definition of $\tilde{L}^1\tilde{u}$ satisfy the constraints (3.15). Hence, the cell problems (3.14) are well-posed and u^3 is well defined. Similarly, combining Lemma 3.4 with Proposition 3.5, we verify that $L^2\tilde{u} = \tilde{L}^2\tilde{u}$ and the definition of $\tilde{L}^2$ ensures that u^4 is well defined. Note that thanks to assumption (1.11), we verify that $x \mapsto \mathcal{B}^\varepsilon\tilde{u}(t,x)$ is Ω-periodic. This proves the existence of the adaptation $\mathcal{B}^\varepsilon\tilde{u}$. By construction (see (3.3)), $\mathcal{B}^\varepsilon\tilde{u}$ satisfies the properties (3.2) and the proof of the theorem is complete.

4. Proof of the main result. In this section, we prove the main result of the paper, Theorem 2.3. The proof is structured as follows. First, we use the correctors introduced in Section 3 to define the adaptation operator $\boldsymbol{\mathcal{B}}^\varepsilon$. This operator satisfies $\boldsymbol{\mathcal{B}}^\varepsilon\tilde{u}(t) = \big[\mathcal{B}^\varepsilon\tilde{u}(t)\big]$, where $\mathcal{B}^\varepsilon\tilde{u}$ is defined in (3.1) and $[\cdot]$ denotes the equivalence class in $\mathcal{W}_{\text{per}}(\Omega) = \mathrm{H}_{\text{per}}^1(\Omega)/\mathbb{R}$. We emphasize that in the proof, we work in the quotient $\mathcal{W}_{\text{per}}(\Omega)$ because $\mathcal{B}^\varepsilon\tilde{u}(t)$ does not have zero mean (alternatively, we can normalize all the non zero mean terms in $\mathcal{B}^\varepsilon\tilde{u}(t)$ and work in $\mathrm{W}_{\text{per}}(\Omega)$). Next, using that $u^\varepsilon - \tilde{u} \in \mathrm{W}_{\text{per}}(\Omega)$, we split the error as

$$\|u^\varepsilon - \tilde{u}\|_{\mathrm{L}^\infty(W)} = \|\big[u^\varepsilon - \tilde{u}\big]\|_{\mathrm{L}^\infty(\mathcal{W})} \le \|\boldsymbol{\mathcal{B}}^\varepsilon\tilde{u} - \big[u^\varepsilon\big]\|_{\mathrm{L}^\infty(\mathcal{W})} + \|\big[\tilde{u}\big] - \boldsymbol{\mathcal{B}}^\varepsilon\tilde{u}\|_{\mathrm{L}^\infty(\mathcal{W})}, \tag{4.1}$$

and estimate both terms. In particular, we prove that $\boldsymbol{\mathcal{B}}^\varepsilon\tilde{u}$ satisfies the same equation as $\big[u^\varepsilon\big]$ up to a remainder of order $\mathcal{O}(\varepsilon^3)$ (Lemma 4.1).

Consider the correctors

$$\chi_i(x), \theta_{ij}^0(x), \theta_i^1(x), \kappa_{ijk}^0(x), \kappa_{ij}^1(x), \kappa_i^2(x), \rho_{ijkl}^0(x), \rho_{ijk}^1(x), \rho_{ij}^2(x), \rho_i^3(x) \in \mathrm{W}_{\text{per}}(Y),$$

defined as the solutions of the cell problems (3.5), (3.8), (3.14) and (3.22), and let $\boldsymbol{\varphi}$ be the solution of (3.26). Propositions 3.3 and 3.5 ensure that $L^1\tilde{u} = \tilde{L}^1\tilde{u}$ and $L^2\tilde{u} = \tilde{L}^2\tilde{u}$, where the definitions of the tensors in $\tilde{L}^1$ (resp. $\tilde{L}^2$) guarantee the well-posedness of the cell problems (3.14) (resp. (3.22)). Let us investigate the regularity of the correctors. Using regularity results for elliptic PDEs (see e.g. [18]), we can prove the following implications, for $n \ge 0$, $m \ge 0$ (see [41, Chapter 6]):

$$\begin{aligned}\chi_i, \theta_{ij}^0, \kappa_{ijk}^0, \rho_{ijkl}^0 \in \mathcal{C}^n(\bar{\Omega}; \mathrm{H}^{m+1}(Y)) \quad &\Leftarrow \quad a \in \mathcal{C}^n(\bar{\Omega}; \mathrm{W}^{m,\infty}(Y)),\\ \theta_i^1, \kappa_{ij}^1, \rho_{ijk}^1 \in \mathcal{C}^n(\bar{\Omega}; \mathrm{H}^{m+1}(Y)) \quad &\Leftarrow \quad a \in \mathcal{C}^n(\bar{\Omega}; \mathrm{W}^{m,\infty}(Y)) \cap \mathcal{C}^{n+1}(\bar{\Omega}; \mathrm{W}^{\{m-1\}_+,\infty}(Y)),\\ \kappa_i^2, \rho_{ij}^2 \in \mathcal{C}^n(\bar{\Omega}; \mathrm{H}^{m+1}(Y)) \quad &\Leftarrow \quad a \in \cap_{k=0}^2 \mathcal{C}^{n+k}(\bar{\Omega}; \mathrm{W}^{\{m-k\}_+,\infty}(Y)),\\ \rho_i^3 \in \mathcal{C}^n(\bar{\Omega}; \mathrm{H}^{m+1}(Y)) \quad &\Leftarrow \quad a \in \cap_{k=0}^3 \mathcal{C}^{n+k}(\bar{\Omega}; \mathrm{W}^{\{m-k\}_+,\infty}(Y)),\end{aligned} \tag{4.2}$$

$$\begin{aligned} a^0_{ij}, \bar{a}^{24}_{ijkl}, b^{22}_{ij} \in \mathcal{C}^n(\bar{\Omega}) \quad &\Leftarrow \quad a \in \mathcal{C}^n(\bar{\Omega}; \mathrm{L}^\infty(Y)), \\ \bar{a}^{12}_{ij} \in \mathcal{C}^n(\bar{\Omega}) \quad &\Leftarrow \quad a \in \mathcal{C}^{n+1}(\bar{\Omega}; \mathrm{L}^\infty(Y)), \\ \bar{a}^{22}_{ij} \in \mathcal{C}^n(\bar{\Omega}) \quad &\Leftarrow \quad a \in \mathcal{C}^{n+2}(\bar{\Omega}; \mathrm{L}^\infty(Y)), \end{aligned}$$

where $\{\cdot\}_+ = \max\{0, \cdot\}$. In particular, under the assumption of Theorem 2.3, all the correctors belong to $\mathcal{C}^1(\bar{\Omega}; \mathrm{H}^3(Y)) \cap \mathcal{C}^2(\bar{\Omega}; \mathrm{H}^2(Y))$. As $d \le 3$, the Sobolev embedding $\mathrm{H}^2_{\mathrm{per}}(Y) \hookrightarrow \mathcal{C}^0_{\mathrm{per}}(\bar{Y})$ holds and the correctors belongs to $\mathcal{C}^1(\bar{\Omega}; \mathcal{C}^1_{\mathrm{per}}(\bar{Y})) \cap \mathcal{C}^2(\bar{\Omega}; \mathcal{C}^0_{\mathrm{per}}(\bar{Y}))$. Hence, the following estimates hold (needed in the proof of Lemma 4.1 below)

$$\begin{aligned} \max_{ijkl} \Big\{ &\|\chi_i\|_{\mathcal{C}^0(\mathcal{C}^0)}, \|\theta^0_{ij}\|_{\mathcal{C}^1(\mathcal{C}^1)}, \|\theta^1_i\|_{\mathcal{C}^0(\mathcal{C}^0)}, \|\kappa^0_{ijk}\|_{\mathcal{C}^2(\mathcal{C}^0)}, \|\kappa^1_{ij}\|_{\mathcal{C}^2(\mathcal{C}^0)}, \\ &\|\kappa^2_i\|_{\mathcal{C}^2(\mathcal{C}^0)}, \|\rho^0_{ijkl}\|_{\mathcal{C}^1(\mathcal{C}^1)}, \|\rho^1_{ijk}\|_{\mathcal{C}^1(\mathcal{C}^1)}, \|\rho^2_{ij}\|_{\mathcal{C}^1(\mathcal{C}^1)}, \|\rho^3_i\|_{\mathcal{C}^1(\mathcal{C}^1)}, \\ &\|\bar{a}^{12}_{ij}\|_{\mathcal{C}^2}, |b^{10}|, \|\bar{a}^{24}_{ijkl}\|_{\mathcal{C}^3}, \|b^{22}_{ij}\|_{\mathcal{C}^2}, \|\bar{a}^{22}_{ij}\|_{\mathcal{C}^2}, |b^{20}| \Big\} \\ &\qquad \le C_1(a, \lambda, Y) + \delta C_2(a, \lambda, Y), \end{aligned} \tag{4.3}$$

where $C_i(a, \lambda, Y)$ depend only on λ, Y, $\|a\|_{\mathcal{C}^1(\mathrm{W}^{2,\infty})}$, $\|a\|_{\mathcal{C}^2(\mathrm{W}^{1,\infty})}$, and $\|a\|_{\mathcal{C}^4(\mathrm{L}^\infty)}$, and δ is the parameter.

Let us introduce the following useful application of the Green formula (see [8] for a proof): for $c \in [\mathrm{W}^{1,\infty}_{\mathrm{per}}(\Omega)]^d$, $v \in \mathrm{H}^1_{\mathrm{per}}(\Omega)$, and $\boldsymbol{w} \in \mathcal{W}_{\mathrm{per}}(\Omega)$, we have

$$\big([c_m \partial_m v], \boldsymbol{w} \big)_{\mathcal{L}^2} = -\big([\partial_m c_m v], \boldsymbol{w} \big)_{\mathcal{L}^2} - \big(c_m, \partial_m \boldsymbol{w} \big)_{\mathrm{L}^2}, \tag{4.4}$$

where we recall that $\partial_m c_m = \sum_{m=1}^d \partial_m c_m$. In order to shorten the notation, we define the following functions of $\mathcal{C}^1_{\mathrm{per}}(\bar{\Omega})$: $\chi^\varepsilon_i = \chi_i\big(\cdot, \frac{\cdot}{\varepsilon}\big)$, $\theta^{0\varepsilon}_{ij} = \theta^0_{ij}\big(\cdot, \frac{\cdot}{\varepsilon}\big)$, $\theta^{1\varepsilon}_i = \theta^1_i\big(\cdot, \frac{\cdot}{\varepsilon}\big)$, and similarly $\kappa^{0\varepsilon}_{ijk}, \kappa^{1\varepsilon}_{ij}, \kappa^{2\varepsilon}_i, \rho^{0\varepsilon}_{ijkl}, \rho^{1\varepsilon}_{ijk}, \rho^{2\varepsilon}_{ij}, \rho^{3\varepsilon}_i$. We define then the operators $\boldsymbol{\mathcal{B}}^\varepsilon_i : \mathrm{H}^3_{\mathrm{per}}(\Omega) \to \mathcal{W}^*_{\mathrm{per}}(\Omega)$ for $v \in \mathrm{H}^3_{\mathrm{per}}(\Omega)$ as

$$\begin{aligned} \big\langle \boldsymbol{\mathcal{B}}^\varepsilon_0 v, \boldsymbol{w} \big\rangle &= \big([v], \boldsymbol{w} \big)_{\mathcal{L}^2}, \\ \big\langle \boldsymbol{\mathcal{B}}^\varepsilon_1 v, \boldsymbol{w} \big\rangle &= \big(\varepsilon [\chi^\varepsilon_i \partial_i v], \boldsymbol{w} \big)_{\mathcal{L}^2}, \\ \big\langle \boldsymbol{\mathcal{B}}^\varepsilon_2 v, \boldsymbol{w} \big\rangle &= \big(\varepsilon^2 [(-\partial_m \theta^{0\varepsilon}_{mi} + \theta^{1\varepsilon}_i) \partial_i v], \boldsymbol{w} \big)_{\mathcal{L}^2} - \big(\varepsilon^2 \theta^{0\varepsilon}_{mi} \partial_i v, \partial_m \boldsymbol{w} \big)_{\mathrm{L}^2}, \\ \big\langle \boldsymbol{\mathcal{B}}^\varepsilon_3 v, \boldsymbol{w} \big\rangle &= \big(\varepsilon^3 [\kappa^{0\varepsilon}_{ijk} \partial^3_{ijk} v + \kappa^{1\varepsilon}_{ij} \partial^2_{ij} v + \kappa^{2\varepsilon}_i \partial_i v], \boldsymbol{w} \big)_{\mathcal{L}^2}, \\ \big\langle \boldsymbol{\mathcal{B}}^\varepsilon_4 v, \boldsymbol{w} \big\rangle &= \big(\varepsilon^4 [(-\partial_m \rho^{0\varepsilon}_{mijk} + \rho^{1\varepsilon}_{ijk}) \partial^3_{ijk} v + \rho^{2\varepsilon}_{ij} \partial^2_{ij} v + \rho^{3\varepsilon}_i \partial_i v], \boldsymbol{w} \big)_{\mathcal{L}^2} - \big(\varepsilon^4 \rho^{0\varepsilon}_{mijk} \partial^3_{ijk} v, \partial_m \boldsymbol{w} \big)_{\mathrm{L}^2}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual evaluation $\langle \cdot, \cdot \rangle_{\mathcal{W}^*_{\mathrm{per}}, \mathcal{W}_{\mathrm{per}}}$. The adaptation operator is then defined as

$$\boldsymbol{\mathcal{B}}^\varepsilon : \mathrm{L}^2(0, T^\varepsilon; \mathrm{H}^3_{\mathrm{per}}(\Omega)) \to \mathrm{L}^2(0, T^\varepsilon; \mathcal{W}^*_{\mathrm{per}}(\Omega)), \quad v \mapsto \boldsymbol{\mathcal{B}}^\varepsilon v(t) = \sum_{i=0}^4 \boldsymbol{\mathcal{B}}^\varepsilon_i (v(t)) + \boldsymbol{\varphi}(t). \tag{4.5}$$

Note that if $v \in \mathrm{L}^2(0, T^\varepsilon; \mathrm{H}^1_{\mathrm{per}}(\Omega) \cap \mathrm{H}^5(\Omega))$, then $\boldsymbol{\mathcal{B}}^\varepsilon v(t) \in \mathcal{W}_{\mathrm{per}}(\Omega)$ and, using (4.4), we verify that $\boldsymbol{\mathcal{B}}^\varepsilon \tilde{u}(t) = [\mathcal{B}^\varepsilon \tilde{u}(t)]$, where $\mathcal{B}^\varepsilon \tilde{u}$ is defined in (3.1) (with $\{u^k\}_{k=1}^4$ defined in (3.4), (3.7), (3.10), and (3.18)). For $\mathcal{A}^\varepsilon = -\nabla_x \cdot \big(a^\varepsilon(x) \nabla_x \cdot \big)$, we thus define

$$\big\langle \mathcal{A}^\varepsilon \boldsymbol{\mathcal{B}}^\varepsilon \tilde{u}(t), \boldsymbol{w} \big\rangle_{\mathcal{W}^*_{\mathrm{per}}, \mathcal{W}_{\mathrm{per}}} = \big\langle \mathcal{A}^\varepsilon [\mathcal{B}^\varepsilon v(t)], \boldsymbol{w} \big\rangle_{\mathcal{W}^*_{\mathrm{per}}, \mathcal{W}_{\mathrm{per}}}.$$

Remark that the definition of $\boldsymbol{\mathcal{B}}^\varepsilon$ in (4.5) allows to consider functions with lower regularity than $\mathcal{B}^\varepsilon$. In particular, $\boldsymbol{\mathcal{B}}^\varepsilon(\partial^2_t \tilde{u})$ is well-defined, as $\partial^2_t \tilde{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^3_{\mathrm{per}}(\Omega))$. This is needed to prove the following lemma.

Lemma 4.1. *Under the assumptions of Theorem 2.3, $\boldsymbol{\mathcal{B}}^\varepsilon \tilde{u}$ satisfies*

$$(\partial^2_t + \mathcal{A}^\varepsilon) \boldsymbol{\mathcal{B}}^\varepsilon \tilde{u}(t) = [f(t)] + \boldsymbol{\mathcal{R}}^\varepsilon \tilde{u}(t) \quad \textit{in } \mathcal{W}^*_{\mathrm{per}}(\Omega) \quad \textit{for a.e. } t \in [0, T^\varepsilon],$$

where the remainder $\boldsymbol{\mathcal{R}}^\varepsilon \tilde{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathcal{W}^*_{\mathrm{per}}(\Omega))$ is given as

$$\big\langle \boldsymbol{\mathcal{R}}^\varepsilon \tilde{u}(t), \boldsymbol{w} \big\rangle_{\mathcal{W}^*_{\mathrm{per}}, \mathcal{W}^*_{\mathrm{per}}} = \big((\boldsymbol{\mathcal{R}}^\varepsilon \tilde{u})_0(t), \boldsymbol{w} \big)_{\mathcal{L}^2} + \big((\boldsymbol{\mathcal{R}}^\varepsilon \tilde{u})_1(t), \nabla_x \boldsymbol{w} \big)_{\mathrm{L}^2},$$

with the bound

$$\begin{aligned}\|(\boldsymbol{\mathcal{R}}^\varepsilon\tilde u)_0\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathcal{L}^2(\Omega))}&+\|(\boldsymbol{\mathcal{R}}^\varepsilon\tilde u)_1\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))}\\ &\leq C\varepsilon^3\Big(\textstyle\sum_{k=1}^5|\tilde u|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^k(\Omega))}+\|\partial_t^2\tilde u\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^3(\Omega))}\Big),\end{aligned}$$

for a constant C that depends only on λ, Y, $\|a\|_{\mathcal{C}^1(\bar\Omega;\mathrm{W}^{2,\infty}(Y))}$, $\|a\|_{\mathcal{C}^2(\bar\Omega;\mathrm{W}^{1,\infty}(Y))}$, $\|a\|_{\mathcal{C}^4(\bar\Omega;\mathrm{L}^\infty(Y))}$, and δ.

Proof. Let us denote $\langle\cdot,\cdot\rangle_{\mathcal{W}_{\mathrm{per}},\mathcal{W}_{\mathrm{per}}}$ as $\langle\cdot,\cdot\rangle$. For a fixed $t\in[0,T^\varepsilon]$, we compute the remainder $\boldsymbol{\mathcal{R}}^\varepsilon\tilde u(t)=(\partial_t^2+\mathcal{A}^\varepsilon)\boldsymbol{\mathcal{B}}^\varepsilon\tilde u(t)-\big[f(t)\big]$. Let us first compute explicitly the first term, $\partial_t^2\boldsymbol{\mathcal{B}}^\varepsilon\tilde u(t)$. For the sake of clarity, we drop the notation of the evaluation in t. From the definition of $\boldsymbol{\mathcal{B}}^\varepsilon$ in (4.5), it holds $\partial_t^2\boldsymbol{\mathcal{B}}^\varepsilon\tilde u=\sum_{i=0}^2\boldsymbol{\mathcal{B}}_i^\varepsilon\partial_t^2\tilde u+\partial_t^2\boldsymbol{\varphi}+\boldsymbol{\mathcal{R}}_1^\varepsilon\tilde u$, where $\boldsymbol{\mathcal{R}}_1^\varepsilon\tilde u=\sum_{i=3}^4\boldsymbol{\mathcal{B}}_i^\varepsilon\partial_t^2\tilde u$, i.e.,

$$\begin{aligned}\big\langle\partial_t^2\boldsymbol{\mathcal{B}}^\varepsilon\tilde u,\boldsymbol{w}\big\rangle=\big(\big[\partial_t^2\tilde u\big],\boldsymbol{w}\big)_{\mathcal{L}^2}&+\big(\big[\varepsilon\chi_i^\varepsilon\partial_i\partial_t^2\tilde u+\varepsilon^2(-\partial_m\theta_{mi}^{0\varepsilon}+\theta_i^{1\varepsilon})\partial_i\partial_t^2\tilde u\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &-\big(\varepsilon^2\theta_{mi}^{0\varepsilon}\partial_i\partial_t^2\tilde u,\partial_m\boldsymbol{w}\big)+\big\langle\partial_t^2\boldsymbol{\varphi}+\boldsymbol{\mathcal{R}}_1^\varepsilon\tilde u,\boldsymbol{w}\big\rangle.\end{aligned}\tag{4.6}$$

We rewrite the three first terms of the right hand side. Note that thanks to the regularity of $\tilde u$ and the effective equation (2.9), we have the following equalities

$$\partial_t^2\tilde u=f+\partial_m(a_{mn}^0\partial_n\tilde u)-\varepsilon\tilde L^1\tilde u-\varepsilon^2\tilde L^2\tilde u\quad\text{in }\mathrm{L}_0^2(\Omega),\tag{4.7}$$

$$\partial_i\partial_t^2\tilde u=\partial_if+\partial_{im}^2(a_{mn}^0\partial_n\tilde u)-\varepsilon\partial_i(\tilde L^1\tilde u)-\varepsilon^2\partial_i(\tilde L^2\tilde u)\quad\text{in }\mathrm{L}^2(\Omega).\tag{4.8}$$

Using (4.7), we rewrite the first term of (4.6) as

$$\begin{aligned}\big[\partial_t^2\tilde u\big]=\big[f\big]&+\big[\partial_m(a_{mn}^0\partial_n\tilde u)+\varepsilon\big(-\tilde L^{1,x}\tilde u-b^{10}\partial_m(a_{mn}^0\partial_n\tilde u)\big)+\varepsilon^2\big(-\tilde L^2\tilde u+b^{10}\tilde L^1\tilde u\big)\big]\\ &+\big[-\varepsilon b^{10}f+\varepsilon^3b^{10}\tilde L^2\tilde u\big],\end{aligned}$$

where $\tilde L^{1,x}=\tilde L^1-b^{10}\partial_t^2$. Using the definitions of $\tilde L^1$ and $\tilde L^2$ and (4.7), we have

$$\begin{aligned}&\varepsilon^2\big(\big[-\tilde L^2\tilde u+b^{10}\tilde L^1\tilde u\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &\quad=\varepsilon^2\big(\big[-\tilde L^{2,x}\tilde u-b^{10}\tilde L^{1,x}\tilde u+((b^{10})^2-b^{20})\partial_m(a_{mn}^0\partial_n\tilde u)\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &\qquad-\varepsilon^2\big(b_{mi}^{22}\partial_i\partial_t^2\tilde u,\partial_m\boldsymbol{w}\big)_{\mathrm{L}^2}+\varepsilon^2\big(\big[((b^{10})^2-b^{20})(f+\varepsilon\tilde L^1\tilde u+\varepsilon^2\tilde L^2\tilde u)\big],\boldsymbol{w}\big)_{\mathcal{L}^2},\end{aligned}$$

where $\tilde L^{2,x}=\tilde L^2+\partial_i(b_{ij}^{22}\partial_j\partial_t^2\cdot)-b^{20}\partial_t^2$. We thus obtain

$$\begin{aligned}\big(\big[\partial_t^2\tilde u\big],\boldsymbol{w}\big)_{\mathcal{L}^2}=\big(\big[f\big]&+\big[\partial_m(a_{mn}^0\partial_n\tilde u)+\varepsilon\big(-\tilde L^{1,x}\tilde u-b^{10}\partial_m(a_{mn}^0\partial_n\tilde u)\big)\\ &+\varepsilon^2\big(-\tilde L^{2,x}\tilde u+b^{10}\tilde L^{1,x}\tilde u+((b^{10})^2-b^{20})\partial_m(a_{mn}^0\partial_n\tilde u)\big],\boldsymbol{w}\big)_{\mathcal{L}^2}\\ &-\big(\varepsilon^2b_{mi}^{22}\partial_i\partial_t^2\tilde u,\partial_m\boldsymbol{w}\big)_{\mathrm{L}^2}+\big(\boldsymbol{\mathcal{S}}_1^\varepsilon f+\boldsymbol{\mathcal{R}}_2^\varepsilon\tilde u,\boldsymbol{w}\big)_{\mathcal{L}^2},\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{S}}_1^\varepsilon f&=\big[-\varepsilon b^{10}f+\varepsilon^2((b^{10})^2-b^{20})f\big],\\ \boldsymbol{\mathcal{R}}_2^\varepsilon\tilde u&=\big[\varepsilon^3b^{10}\tilde L^2\tilde u+\varepsilon^3((b^{10})^2-b^{20})(\tilde L^1\tilde u+\varepsilon\tilde L^2\tilde u)\big].\end{aligned}$$

Next, we use (4.8) and then (4.7) to write the second term of (4.6) as

$$\big[\varepsilon\chi_i^\varepsilon\partial_i\partial_t^2\tilde u\big]=\big[\varepsilon\chi_i^\varepsilon\partial_{im}^2(a_{mn}^0\partial_n\tilde u)-\varepsilon^2\chi_i^\varepsilon\partial_i(\tilde L^{1,x}\tilde u)-\varepsilon^2\chi_i^\varepsilon\partial_i(b^{10}\partial_m(a_{mn}^0\partial_n\tilde u))\big]+\boldsymbol{\mathcal{S}}_2^\varepsilon f+\boldsymbol{\mathcal{R}}_3^\varepsilon\tilde u,\tag{4.10}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{S}}_2^\varepsilon f&=\big[\varepsilon\chi_i^\varepsilon\partial_if-\varepsilon^2\chi_i^\varepsilon\partial_i(b^{10}f)\big],\\ \boldsymbol{\mathcal{R}}_3^\varepsilon\tilde u&=\big[-\varepsilon^3\chi_i^\varepsilon\partial_i(\tilde L^2\tilde u)+\varepsilon^3\chi_i^\varepsilon\partial_i(b^{10}(\tilde L^1\tilde u+\varepsilon\tilde L^2\tilde u))\big].\end{aligned}$$

Furthermore, using (4.8) and formula (4.4), we rewrite

$$\begin{aligned}&-\big(\big[\varepsilon^2\partial_m\theta_{mi}^{0\varepsilon}\partial_i\partial_t^2\tilde u\big],\boldsymbol{w}\big)_{\mathcal{L}^2}-\big(\varepsilon^2(\theta_{mi}^{0\varepsilon}+b_{mi}^{22})\partial_i\partial_t^2\tilde u,\partial_m\boldsymbol{w}\big)_{\mathrm{L}^2}\\ &\quad=\big(\big[\varepsilon^2\theta_{ij}^{0\varepsilon}\partial_{ijm}^3(a_{mn}^0\partial_n\tilde u)+\partial_i(b_{ij}^{22}\partial_{jm}^2(a_{mn}^0\partial_n\tilde u))\big],\boldsymbol{w}\big)_{\mathcal{L}^2}+\big\langle\boldsymbol{\mathcal{S}}_3^\varepsilon f+\boldsymbol{\mathcal{R}}_4^\varepsilon\tilde u,\boldsymbol{w}\big\rangle.\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}
&\left\langle \boldsymbol{\mathcal{S}}_3^\varepsilon f, \boldsymbol{w} \right\rangle = \left(\varepsilon^2 \left[\theta_{ij}^{0\varepsilon} \partial_{ij}^2 f + \partial_i(b_{ij}^{22} \partial_j f)\right], \boldsymbol{w}\right)_{\mathcal{L}^2},\\
&\left\langle \boldsymbol{\mathcal{R}}_4^\varepsilon \tilde{u}, \boldsymbol{w} \right\rangle = \left(\varepsilon^3 \left[\partial_m \theta_{mi}^{0\varepsilon} \partial_i(\tilde{L}^1 \tilde{u} + \varepsilon \tilde{L}^2 \tilde{u})\right], \boldsymbol{w}\right)_{\mathcal{L}^2} + \left(\varepsilon^3 (\theta_{mi}^{0\varepsilon} + b_{mi}^{22}) \partial_i(\tilde{L}^1 \tilde{u} + \varepsilon \tilde{L}^2 \tilde{u}), \partial_m \boldsymbol{w}\right)_{\mathrm{L}^2},
\end{aligned}$$

and, using (4.8), we rewrite

$$\left[\varepsilon^2 \theta_i^{1\varepsilon} \partial_i \partial_t^2 \tilde{u}\right] = \left[\varepsilon^2 \theta_i^{1\varepsilon} \partial_{im}^2 (a_{mn}^0 \partial_n \tilde{u})\right] + \boldsymbol{\mathcal{S}}_4^\varepsilon f + \boldsymbol{\mathcal{R}}_5^\varepsilon \tilde{u}, \tag{4.12}$$

where $\boldsymbol{\mathcal{S}}_4^\varepsilon f = \left[\varepsilon^2 \theta_i^{1\varepsilon} \partial_i f\right]$ and $\boldsymbol{\mathcal{R}}_5^\varepsilon \tilde{u} = \left[\varepsilon^3 \theta_i^{1\varepsilon} \partial_i(\tilde{L}^1 \tilde{u} + \varepsilon \tilde{L}^2 \tilde{u})\right]$. Combining equalities (4.6), (4.9), (4.10), (4.11) and (4.12), we finally obtain

$$\begin{aligned}
\partial_t^2 \boldsymbol{\mathcal{B}}^\varepsilon \tilde{u} = {} & [f] + \left[\partial_m (a_{mn}^0 \partial_n \tilde{u})\right] + \varepsilon \left[\chi_i \partial_{im}^2 (a_{mn}^0 \partial_n \tilde{u}) - \tilde{L}^{1,x} \tilde{u} - b^{10} \partial_m (a_{mn}^0 \partial_n \tilde{u})\right]\\
& + \varepsilon^2 \big[\theta_{ij}^{0\varepsilon} \partial_{ijm}^3 (a_{mn}^0 \partial_n \tilde{u}) + \theta_i^{1\varepsilon} \partial_{im}^2 (a_{mn}^0 \partial_n \tilde{u}) - \chi_i^\varepsilon \partial_i (\tilde{L}^{1,x} \tilde{u}) - \chi_i^\varepsilon \partial_i (b^{10} \partial_m (a_{mn}^0 \partial_n \tilde{u}))\\
& \quad - \tilde{L}^{2,x} \tilde{u} + b^{10} \tilde{L}^{1,x} \tilde{u} + ((b^{10})^2 - b^{20}) \partial_m (a_{mn}^0 \partial_n \tilde{u}) + \partial_i (b_{ij}^{22} \partial_{jm}^2 (a_{mn}^0 \partial_n \tilde{u}))\big]\\
& + \partial_t^2 \boldsymbol{\varphi} + \textstyle\sum_{i=1}^4 \boldsymbol{\mathcal{S}}_i^\varepsilon f + \sum_{i=1}^5 \boldsymbol{\mathcal{R}}_i^\varepsilon \tilde{u}.
\end{aligned} \tag{4.13}$$

For the second term, $\mathcal{A}^\varepsilon \boldsymbol{\mathcal{B}}^\varepsilon \tilde{u}(t)$, we have (the correctors and a are evaluated at $\left(x, y = \frac{x}{\varepsilon}\right)$)

$$\begin{aligned}
& \mathcal{A}^\varepsilon \boldsymbol{\mathcal{B}}^\varepsilon \tilde{u} =\\
\big[\quad & \varepsilon^{-1} \big(- \nabla_y \cdot (a(\nabla_y \chi_i + e_i)) \big) \partial_i \tilde{u}\\
& + \varepsilon^0 \quad \big(- \nabla_y \cdot (a(\nabla_y \theta_{ij}^0 + e_i \chi_j)) - e_i^T a(\nabla_y \chi_j + e_j) \big) \partial_{ij}^2 \tilde{u}\\
& + \varepsilon^0 \quad \big(- \nabla_y \cdot (a(\nabla_y \theta_i^1 + \nabla_x \chi_i)) - \nabla_x \cdot (a(\nabla_y \chi_i + e_i)) \big) \partial_i \tilde{u}\\
& + \varepsilon^1 \quad \big(- \nabla_y \cdot (a(\nabla_y \kappa_{ijk}^0 + e_i \theta_{jk}^0)) - e_i^T a(\nabla_y \theta_{jk}^0 + e_j \chi_k) \big) \partial_{ijk}^3 \tilde{u}\\
& + \varepsilon^1 \quad \big(- \nabla_y \cdot (a(\nabla_y \kappa_{ij}^1 + \nabla_x \theta_{ij}^0 + e_i \theta_j^1)) - \nabla_x \cdot (a(\nabla_y \theta_{ij}^0 + e_i \chi_j)) - e_i^T a(\nabla_y \theta_j^1 + \nabla_x \chi_j) \big) \partial_{ij}^2 \tilde{u}\\
& + \varepsilon^1 \quad \big(- \nabla_y \cdot (a(\nabla_y \kappa_i^2 + \nabla_x \theta_i^1)) - \nabla_x \cdot (a(\nabla_y \theta_i^1 + \nabla_x \chi_i)) \big) \partial_i \tilde{u}\\
& + \varepsilon^2 \quad \big(- \nabla_y \cdot (a(\nabla_y \rho_{ijkl}^0 + e_i \kappa_{jkl}^0)) - e_i^T a(\nabla_y \kappa_{jkl}^0 + e_j \theta_{kl}^0) \big) \partial_{ijkl}^4 \tilde{u}\\
& + \varepsilon^2 \quad \big(- \nabla_y \cdot (a(\nabla_y \rho_{ijk}^1 + \nabla_x \kappa_{ijk}^0 + e_i \kappa_{jk}^1)) - \nabla_x \cdot (a(\nabla_y \kappa_{ijk}^0 + e_i \theta_{jk}^0))\\
& \qquad\qquad\qquad\qquad\qquad - e_i^T a(\nabla_y \kappa_{jk}^1 + \nabla_x \theta_{jk}^0) \big) \partial_{ijk}^3 \tilde{u}\\
& + \varepsilon^2 \quad \big(- \nabla_y \cdot (a(\nabla_y \rho_{ij}^2 + \nabla_x \kappa_{ij}^1 + e_i \kappa_j^2)) - \nabla_x \cdot (a(\nabla_y \kappa_{ij}^1 + \nabla_x \theta_{ij}^0 + e_i \theta_j^1))\\
& \qquad\qquad\qquad\qquad\qquad - e_i^T a(\nabla_y \kappa_j^2 + \nabla_x \theta_j^1) \big) \partial_{ij}^2 \tilde{u}\\
& + \varepsilon^2 \quad \big(- \nabla_y \cdot (a(\nabla_y \rho_i^3 + \nabla_x \kappa_i^2)) - \nabla_x (a(\nabla_y \kappa_i^2 + \nabla_x \theta_i^1)) \big) \partial_i \tilde{u} \qquad \big]\\
& + \mathcal{A}^\varepsilon \boldsymbol{\varphi} + \boldsymbol{\mathcal{R}}_6^\varepsilon \tilde{u} + \boldsymbol{\mathcal{R}}_7^\varepsilon \tilde{u},
\end{aligned} \tag{4.14}$$

where, defining the following functions of (x, y),

$$\begin{aligned}
R_{ijkl}^0 &= a(\nabla_y \rho_{ijkl}^0 + e_i \kappa_{jkl}^0), & R_{ijk}^1 &= a(\nabla_y \rho_{ijk}^1 + \nabla_x \kappa_{ijk}^0 + e_i \kappa_{jk}^1),\\
R_{ij}^2 &= a(\nabla_y \rho_{ij}^2 + \nabla_x \kappa_{ij}^1 + e_i \kappa_j^2), & R_i^3 &= a(\nabla_y \rho_i^3 + \nabla_x \kappa_i^2),
\end{aligned}$$

the remainders $\boldsymbol{\mathcal{R}}_6^\varepsilon \tilde{u}$ and $\boldsymbol{\mathcal{R}}_7^\varepsilon \tilde{u}$ are given by

$$\begin{aligned}
\boldsymbol{\mathcal{R}}_6^\varepsilon \tilde{u} &= \varepsilon^3 \big[e_m^T R_{ijkl}^0 \partial_{mijkl}^5 \tilde{u} + \nabla_x \cdot R_{ijkl}^0 \partial_{ijkl}^4 \tilde{u} + e_m^T R_{ijk}^1 \partial_{mijk}^4 \tilde{u} + \nabla_x \cdot R_{ijk}^1 \partial_{ijk}^3 \tilde{u}\\
&\qquad + e_m^T R_{ij}^2 \partial_{mij}^3 \tilde{u} + \nabla_x \cdot R_{ij}^2 \partial_{ij}^2 \tilde{u} + e_m^T R_i^3 \partial_{mi}^2 \tilde{u} + \nabla_x \cdot R_i^3 \partial_i \tilde{u}\big],\\
\left\langle \boldsymbol{\mathcal{R}}_7^\varepsilon \tilde{u}, \boldsymbol{w} \right\rangle &= \varepsilon^4 \big(e_m^T a (\nabla_x \rho_{ijkl}^0 \partial_{ijkl}^4 \tilde{u} + \nabla_x \rho_{ijk}^1 \partial_{ijk}^3 \tilde{u} + \nabla_x \rho_{ij}^2 \partial_{ij}^2 \tilde{u} + \nabla_x \rho_i^3 \partial_i \tilde{u}, \partial_m \boldsymbol{w}\big)_{\mathrm{L}^2}\\
&\qquad + \varepsilon^4 \big(a_{mn} \rho_{ijkl}^0 \partial_{nijkl}^5 \tilde{u} + a_{mn} \rho_{ijk}^1 \partial_{nijk}^4 \tilde{u} + a_{mn} \rho_{ij}^2 \partial_{nij}^3 \tilde{u} + a_{mn} \rho_i^3 \partial_{ni}^2 \tilde{u}, \partial_m \boldsymbol{w}\big)_{\mathrm{L}^2}.
\end{aligned}$$

Combining now (4.13) and (4.14), and using cell problems (3.5), (3.8), (3.14), (3.22), and the definition of $\boldsymbol{\varphi}$ in (3.26) (verify that $\sum_{i=1}^4 \boldsymbol{\mathcal{S}}_i^\varepsilon f = \boldsymbol{\mathcal{S}}^\varepsilon f$), we obtain the remainder $\boldsymbol{\mathcal{R}}^\varepsilon \tilde{u} = \sum_{i=1}^7 \boldsymbol{\mathcal{R}}_i^\varepsilon \tilde{u}$. Using (4.3),we verify that $\boldsymbol{\mathcal{R}}^\varepsilon \tilde{u}$ satisfies estimate (4.6) and the proof of the lemma is complete. ∎

Let us recall the following error estimate, proved in [8].

LEMMA 4.2. *Assume that* $\boldsymbol{\eta} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathcal{W}_{\mathrm{per}}(\Omega))$, *with* $\partial_t\boldsymbol{\eta} \in \mathrm{L}^\infty(0,T^\varepsilon;\mathcal{L}^2(\Omega))$, $\partial_t^2\boldsymbol{\eta} \in \mathrm{L}^2(0,T^\varepsilon;\mathcal{W}^*_{\mathrm{per}}(\Omega))$ *satisfies*

$$\begin{aligned} &\partial_t^2\boldsymbol{\eta}(t) + \mathcal{A}^\varepsilon\boldsymbol{\eta}(t) = \boldsymbol{r}(t) \quad \text{in } \mathcal{W}^*_{\mathrm{per}}(\Omega) \quad \text{for a.e. } t\in[0,T^\varepsilon],\\ &\boldsymbol{\eta}(0) = \boldsymbol{\eta}^0, \quad \partial_t\boldsymbol{\eta}(0) = \boldsymbol{\eta}^1, \end{aligned} \tag{4.15}$$

where $\boldsymbol{\eta}^0 \in \mathcal{W}_{\mathrm{per}}(\Omega)$, $\boldsymbol{\eta}^1 \in \mathcal{L}^2(\Omega)$, *and* $\boldsymbol{r} \in \mathrm{L}^2(0,T^\varepsilon;\mathcal{W}^*_{\mathrm{per}}(\Omega))$ *is given as*

$$\langle \boldsymbol{r}(t),\boldsymbol{w}\rangle_{\mathcal{W}^*_{\mathrm{per}}(\Omega),\mathcal{W}_{\mathrm{per}}(\Omega)} = \big(\boldsymbol{r}_0(t),\boldsymbol{w}\big)_{\mathcal{L}^2(\Omega)} + \big(r_1(t),\nabla_x\boldsymbol{w}\big)_{\mathrm{L}^2(\Omega)},$$

with $\boldsymbol{r}_0 \in \mathrm{L}^2(0,T^\varepsilon;\mathcal{L}^2(\Omega))$ *and* $r_1 \in [\mathrm{L}^2(0,T^\varepsilon;\mathrm{L}^2(\Omega))]^d$. *Then the following estimate holds*

$$\|\boldsymbol{\eta}\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathcal{W})} \le C(\lambda)\Big(\|\boldsymbol{\eta}^1\|_{\mathcal{L}^2(\Omega)} + \|\boldsymbol{\eta}^0\|_{\mathcal{L}^2(\Omega)} + \varepsilon^{-2}T\big(\|\boldsymbol{r}_0\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathcal{L}^2(\Omega))} + \|r_1\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))}\big)\Big), \tag{4.16}$$

where $C(\lambda)$ *depends only on the ellipticity constant* λ *and the norm* $\|\cdot\|_{\mathcal{W}}$ *is defined in* (1.7).

We now have all the tool to prove the theorem.

Proof of Theorem 2.3.

Let us estimate the two terms of the right hand side in (4.1). First, note that $\boldsymbol{\eta} = [\![u^\varepsilon]\!] - \boldsymbol{\mathcal{B}}^\varepsilon\tilde{u}$ satisfies $(\partial_t^2 + \mathcal{A}^\varepsilon)\boldsymbol{\eta}(t) = \boldsymbol{\mathcal{R}}^\varepsilon\tilde{u}(t)$ in $\mathcal{W}^*_{\mathrm{per}}(\Omega)$ for a.e $t\in[0,T^\varepsilon]$, where $\boldsymbol{\mathcal{R}}^\varepsilon\tilde{u}$ is defined in Lemma 4.1. Hence, using Lemma 4.2, the first term in (4.1) satisfies

$$\|[\![u^\varepsilon]\!] - \boldsymbol{\mathcal{B}}^\varepsilon\tilde{u}\|_{\mathrm{L}^\infty(\mathcal{W})} \le C\varepsilon\Big(\|g^1\|_{\mathrm{H}^4} + \|g^0\|_{\mathrm{H}^4} + \textstyle\sum_{k=1}^5 |\tilde{u}|_{\mathrm{L}^\infty(\mathrm{H}^k)} + \|\partial_t^2\tilde{u}\|_{\mathrm{L}^\infty(\mathrm{H}^3)}\Big), \tag{4.17}$$

where C depends on T, λ, Y, $\|a\|_{\mathcal{C}^1(\bar\Omega;\mathrm{W}^{2,\infty}(Y))}$, $\|a\|_{\mathcal{C}^2(\bar\Omega;\mathrm{W}^{1,\infty}(Y))}$, $\|a\|_{\mathcal{C}^4(\bar\Omega;\mathrm{L}^\infty(Y))}$, and δ. Next, using the definition of $\boldsymbol{\mathcal{B}}^\varepsilon$ (4.5) and the estimates (3.27) and (4.3), the second term of (4.1) satisfies

$$\|\boldsymbol{\mathcal{B}}^\varepsilon\tilde{u} - [\![\tilde{u}]\!]\|_{\mathrm{L}^\infty(\mathcal{W})} \le C\varepsilon\Big(\textstyle\sum_{k=1}^5 |\tilde{u}|_{\mathrm{L}^\infty(\mathrm{H}^k)} + \|f\|_{\mathrm{L}^1(\mathrm{H}^2)}\Big), \tag{4.18}$$

where C depends on λ, Y, $\|a\|_{\mathcal{C}^1(\bar\Omega;\mathrm{W}^{2,\infty}(Y))}$, $\|a\|_{\mathcal{C}^2(\bar\Omega;\mathrm{W}^{1,\infty}(Y))}$, $\|a\|_{\mathcal{C}^4(\bar\Omega;\mathrm{L}^\infty(Y))}$, and δ. Combining (4.1), (4.17), and (4.18), we obtain (2.10) and the proof of the theorem is complete. □

5. A priori error analysis of the FE-HMM-L in locally periodic media. The FE-HMM-L is a numerical homogenization method for the long time approximation of the wave equation introduced in [5, 4]. In [9], a priori estimates for the long time error between u^ε and the approximation u_H were proved in one dimension for uniformly periodic tensors. In this section, thanks to Theorem 2.3, we provide a complement to this analysis as we present error estimates that hold in the locally periodic case (again in one dimension). This result is valid in small domains. In addition, we prove a new a priori error estimate that holds in arbitrarily large domain.

Let us first express the family of effective equation in the specific one-dimensional case. Let $x\in\Omega$ be fixed and recall that $\langle\chi(x)\rangle_Y = 0$. As $a(x,\cdot)(1+\partial_y\chi(x,\cdot)) \in \mathrm{H}(\mathrm{div},Y)$, using integration by parts and equation (2.1a), we obtain for any $y_1,y_2\in Y$,

$$a(x,y)(\partial_y\chi(x,y)+1)\Big|_{y=y_1}^{y_2} = -\int_Y \big(H_{y_2}(y) - H_{y_1}(y)\big)\partial_y\big(a(x,y)(\partial_y\chi(x,y)+1)\big)\mathrm{d}y = 0,$$

where H_y is the Heaviside step function centered in y. Hence, the function $y\mapsto a(x,y)(\partial_y\chi(x,y)+1)$ is constant. The definition of a^0 in (2.2) then implies that

$$a(x,y)(\partial_y\chi(x,y)+1) = a^0(x) \quad \forall (x,y)\in\Omega\times Y. \tag{5.1}$$

A similar argument, using (2.1b), (2.1c), and the fact that $a^0(x) = 1/\langle 1/a(x,\cdot)\rangle_Y$, leads to

$$a(x,y)\big(\partial_y\theta^0(x,y) + \chi(x,y)\big) = 0, \qquad a(x,y)\big(\partial_y\theta^1(x,y) + \partial_x\chi(x,y)\big) = 0 \qquad \forall (x,y)\in\Omega\times Y. \tag{5.2}$$

Thanks to (5.1), we verify that the coefficients defined in (2.4) satisfy $p^{13}(x) = q^{12}(x) = 0$ and thus $\bar{a}^{12}(x) = b^{10} = 0$. Similarly, using (5.2) in (2.6) and (2.8), we verify that $\check{a}^{24}(x) = p^{23}(x) = p^{22}(x) = 0$. Hence, in one dimension, the family $\mathcal{E}$ (Definition 2.1) is constituted of the equations

$$\partial_t^2 \tilde{u} - \partial_x(a^0\partial_x \tilde{u}) + \varepsilon^2\big(\partial_x^2(a^{24}\partial_x^2\tilde{u}) - \partial_x(b^{22}\partial_x\partial_t^2\tilde{u}) - \partial_x(a^{22}\partial_x\tilde{u}) + b^{20}\partial_t^2\tilde{u}\big) = f,$$

where the coefficients are defined for some parameter $r \geq 0$ as

$$\begin{aligned} &a^{24}(x) = ra^0(x)^2, && b^{22}(x) = \big\langle \chi(x)^2\big\rangle_Y + ra^0(x), \\ &b^{20} = r\max_{x\in\Omega}\{\partial_x^2 a^0(x)\}, && a^{22}(x) = -ra^0(x)\partial_x^2 a^0(x) + b^{20}a^0(x). \end{aligned} \tag{5.3}$$

In particular, the equation corresponding to the choice $r = 0$ involves the single correction $-\varepsilon^2\partial_x(b^{22}\partial_x\partial_t^2\cdot)$. This is precisely the effective model on which the FE-HMM-L relies (see [5, 4, 9]).

Let us briefly recall the definition of the FE-HMM-L. Let $\mathcal{T}_H$ be a partition of Ω of size H. For $\ell \in \mathbb{N}_{>0}$, the macro finite element space is defined as

$$V_H(\Omega) = \{v_H \in \mathrm{W}_{\mathrm{per}}(\Omega) : v_H|_K \in \mathcal{P}^\ell(K)\ \forall K \in \mathcal{T}_H\}, \tag{5.4}$$

where $\mathcal{P}^\ell(K)$ is the space of polynomials on K of degree at most ℓ. Let $\{\hat{\omega}_j, \hat{x}_j\}_{j=1}^J$ and $\{\hat{\omega}'_j, \hat{x}'_j\}_{j=1}^{J'}$ be the quadrature formulas used for the construction of the stiffness and mass matrices, respectively. We assume that these formulas satisfy the requirements that ensure the optimal convergence rates of the FEM with numerical quadrature (see [21, 20] or [1]). For every macro element $K \in \mathcal{T}_H$ and every $j \in \{1, \ldots, J\}$, we define the sampling domain $K_{\delta j} = x_{K_j} + \delta Y$, where $\delta \geq \varepsilon$. Each sampling domain $K_{\delta j}$ is discretized into a partition $\mathcal{T}_h$ of size h. For $q \in \mathbb{N}_{>0}$, the micro finite element space is defined as

$$V_h(K_{\delta j}) = \{z_h \in \mathrm{W}_{\mathrm{per}}(K_{\delta j}) : z_h|_Q \in \mathcal{P}^q(Q)\ \forall Q \in \mathcal{T}_h\}. \tag{5.5}$$

We define the following bilinear forms: for $v_H, w_H \in V_H(\Omega)$,

$$A_H(v_H, w_H) = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta j}|}\int_{K_{\delta j}} a\big(x_{K_j}, \tfrac{x}{\varepsilon}\big)\partial_x v_{h,K_j}(x)\partial_x w_{h,K_j}(x)\,\mathrm{d}x, \tag{5.6}$$

$$\big(v_H, w_H\big)_Q = \big(v_H, w_H\big)_H + \big(v_H, w_H\big)_M, \tag{5.7}$$

$$\big(v_H, w_H\big)_H = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^{J'} \omega'_{K_j} v_H(x'_{K_j}) w_H(x'_{K_j}), \tag{5.8}$$

$$\big(v_H, w_H\big)_M = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta j}|}\int_{K_{\delta j}} \big(v_{h,K_j} - v_{H,K_j}^{\mathrm{lin}}\big)\big(w_{h,K_j} - w_{H,K_j}^{\mathrm{lin}}\big)(x)\,\mathrm{d}x, \tag{5.9}$$

where the piecewise linear approximation of v_H (resp. w_H) around x_{K_j} is given by

$$v_{H,K_j}^{\mathrm{lin}}(x) = v_H(x_{K_j}) + (x - x_{K_j})\partial_x v_H(x_{K_j}),$$

and the micro functions v_{h,K_j} for v_H (resp. w_H) are the solutions of the following micro problems in $K_{\delta j}$: find v_{h,K_j} such that $(v_{h,K_j} - v_{H,K_j}^{\mathrm{lin}}) \in V_h(K_{\delta j})$ and

$$\big(a\big(x_{K_j}, \tfrac{\cdot}{\varepsilon}\big)\partial_x v_{h,K_j}, \partial_x z_h\big)_{\mathrm{L}^2(K_{\delta j})} = 0 \qquad \forall z_h \in V_h(K_{\delta j}). \tag{5.10}$$

We emphasize that in (5.6) and (5.10), the tensor is collocated in the slow variable, i.e., $a^\varepsilon(x) = a\big(x_{K_j}, \frac{x}{\varepsilon}\big)$ $\forall x \in K_{\delta j}$. The approximation of the FE-HMM-L is $u_H : [0, T^\varepsilon] \to V_H(\Omega)$ such that

$$\begin{aligned} &\big(\partial_t^2 u_H(t), v_H\big)_Q + A_H\big(u_H(t), v_H\big) = \big(f(t), v_H\big)_{\mathrm{L}^2} \quad \forall v_H \in V_H(\Omega) \quad \text{ for a.e. } t \in [0, T^\varepsilon], \\ &u_H(0) = I_H g^0, \quad \partial_t u_H(0) = I_H g^1, \end{aligned} \tag{5.11}$$

where I_H is an interpolation operator onto $V_H(\Omega)$ satisfying the optimal convergence rates (e.g. the nodal interpolation operator defined in [20]).

Let us now combine Theorem 2.3 with the a priori error analysis from [9]. Assume that $\delta/\varepsilon \in \mathbb{N}_{>0}$, $q = 1$, and that the data are sufficiently regular. Then the following a priori error estimate holds:

$$\|u^\varepsilon - u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C\bigg(\varepsilon + \Big(\frac{h}{\varepsilon^2}\Big)^2 + \frac{H^{\ell+1}}{\varepsilon^2} + \frac{H^\ell}{\varepsilon}\bigg) \sum_{k=0}^{4} \|\partial_t^k \bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})}, \tag{5.12}$$

where C is independent of H, h, ε, and δ but depends on Ω.

Note that the dependence of the constant on the domain Ω is an issue when the method is used in pseudoinfinite domains. Indeed, as we consider timescales $\mathcal{O}(\varepsilon^{-2})$, a pseudoinfinite domain must have a diameter of order $\mathcal{O}(\varepsilon^{-2})$ (if the homogenized wave speed is of order $\mathcal{O}(1)$) and (5.12) can not be used. This issue is settled by the following new result.

Theorem 5.1. *Let $\bar{u}$ denote the effective solution in the family that corresponds to the parameter $r = 0$ in (5.3). Assume that $\delta/\varepsilon \in \mathbb{N}_{>0}$, $h \leq \varepsilon$, and $q = 1$. If $a \in \mathcal{C}^\ell(\bar{\Omega}; \mathrm{L}^\infty(Y)) \cap \mathcal{C}^0(\bar{\Omega}; \mathrm{W}^{1,\infty}(Y))$ and $\partial_t^k \bar{u} \in \mathrm{L}^\infty(0, T^\varepsilon; \mathrm{H}^{\ell+1}(\Omega))$ for $0 \leq k \leq 4$, then the error $e = \bar{u} - u_H$ satisfies the estimate*

$$\|\partial_t e\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \|\nabla_x e\|_{\mathrm{L}^\infty(\mathrm{L}^2)} \leq C\bigg(\mathrm{e}_{\mathrm{H}^1}^{\mathrm{data}} + \Big(\frac{h}{\varepsilon^2}\Big)^2 + \frac{H^\ell}{\varepsilon^2}\bigg)\bigg(\sum_{\sigma=1}^{\ell+1} |\bar{u}|_{\mathrm{L}^\infty(\mathrm{H}^\sigma)} + \sum_{k=1}^{4} \|\partial_t^k \bar{u}\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})}\bigg), \tag{5.13}$$

where $\mathrm{e}_{\mathrm{H}^1}^{\mathrm{data}} = |g^0 - I_H g^0|_{\mathrm{H}^1(\Omega)} + \|g^1 - I_H g^1\|_{\mathrm{H}^1(\Omega)}$ and $C = \tilde{C}\big(\|a\|_{\mathcal{C}^\ell(\mathrm{L}^\infty)} + \|a\|_{\mathcal{C}^0(\mathrm{W}^{1,\infty})}\big)$ with $\tilde{C}$ independent of ε, H and Ω.

We emphasize that the constant $\tilde{C}$ is independent of Ω. Hence, combining Theorems 2.3 and 5.1, we obtain (if the data are sufficiently regular and have a spatial support of order $\mathcal{O}(1)$)

$$\|u^\varepsilon - u_H\|_{\mathrm{L}^\infty(0,T^\varepsilon;W)} = \mathcal{O}\bigg(\varepsilon + \Big(\frac{h}{\varepsilon^2}\Big)^2 + \frac{H^\ell}{\varepsilon^2}\bigg), \tag{5.14}$$

where the norm $\|\cdot\|_W$ is defined in (1.8). In particular, estimate (5.14) can be used in pseudoinfinite domains.

To discuss the computational cost of the FE-HMM-L, let us compare it with a standard P1-FEM approximation of u^ε, denoted u_h. As argued in [9] (see also [7]), the optimal error estimate for u_h is given by (if the initial data are well-prepared)

$$\|u^\varepsilon - u_h\|_{\mathrm{L}^\infty(0,T^\varepsilon;\mathrm{L}^2(\Omega))} \leq C\frac{h}{\varepsilon^3}. \tag{5.15}$$

Note that in (5.15), the constant C depends on the domain (through the Poincaré constant and the constant in the elliptic regularity estimate). Hence, in a large domain of diameter $\mathcal{O}(\varepsilon^{-2})$, (5.15) scales worse with respect to ε. On the contrary, we emphasize that (5.14) holds independently of the domain. For the sake of comparison, let us then consider a small domain of diameter $\mathcal{O}(1)$. We fix an order of tolerance τ for the error and, based on the corresponding error estimate, we compute the cost of each method. Let us denote $\mathrm{cost}(\Delta t, N)$ the cost per time-step of the time integration of a second order ODE of dimension N. From (5.15), it follows that the cost of the P1-FEM is $\mathrm{cost}(\Delta t, \varepsilon^{-3}\tau^{-1})$. The cost of the FE-HMM-L, based on (5.14), is $\mathrm{cost}(\Delta t, \varepsilon^{-2/\ell}\tau^{-1/\ell})$, to which we add the offline cost of the resolution of the micro problems, $H^{-1}(\varepsilon/h) = \varepsilon^{-1-2/\ell}\tau^{-1/2-1/\ell}$. Hence, the FE-HMM-L offers a significant reduction of the computational cost. Furthermore, note that in the FE-HMM-L, the macro FE degree ℓ can be increased to reduce the cost. This is not the case for the fine scale FEM as higher order FE do not improve (5.15) (negative powers of ε appear from the higher derivatives of u^ε).

Let us prove Theorem 5.1 (a detailed proof is provided in [41, Chapter 7]). We follow the technique of elliptic projection as done in [27, 14, 15]. We split the error as

$$\bar{u} - u_H = (\bar{u} - \pi_H \bar{u}) - (u_H - \pi_H \bar{u}) = \eta - \zeta_H, \tag{5.16}$$

where $\pi_H \bar{u}$ is a new elliptic projection. The following definition of $\pi_H \bar{u} : [0, T^\varepsilon] \to V_H(\Omega)$ is the key to avoid the dependence of the constant on Ω: for a.e. $t \in [0, T^\varepsilon]$, $\pi_H \bar{u}(t) \in V_H(\Omega)$ satisfies

$$\big(\pi_H \bar{u}(t), v_H\big)_Q + A_H\big(\pi_H \bar{u}(t), v_H\big) = \big(f(t), v_H\big)_{\mathrm{L}^2} - \big(I_H \partial_t^2 \bar{u}(t), v_H\big)_Q + \big(I_H \bar{u}(t), v_H\big)_Q, \tag{5.17}$$

for all test functions $v_H \in V_H(\Omega)$. Using the same technique as in [9], we prove the following lemma.

LEMMA 5.2. *Assume that $\partial_t^k \bar u, \partial_t^{k+2}\bar u \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^{\ell+1}(\Omega))$ for $k \geq 0$. Then $\partial_t^k \pi_H \bar u \in \mathrm{L}^\infty(0,T^\varepsilon;\mathrm{H}^1(\Omega))$ and $\eta = \bar u - \pi_H \bar u$ satisfies*

$$\begin{aligned}&\|I_H \partial_t^k \eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|\partial_t^k \eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \\ &\qquad \leq C\Big(\big((h/\varepsilon)^2 + H^\ell\big)\textstyle\sum_{\sigma=1}^{\ell+1} |\partial_t^k \bar u|_{\mathrm{L}^\infty(\mathrm{H}^\sigma)} + \big(\varepsilon(h/\varepsilon)^2 + H^\ell\big)\|\partial_t^{k+2}\bar u\|_{\mathrm{L}^\infty(\mathrm{H}^{\ell+1})}\Big),\end{aligned} \tag{5.18}$$

where $C = \tilde C\big(\|a\|_{\mathcal{C}^\ell(\mathrm{L}^\infty)} + \|a\|_{\mathcal{C}^0(\mathrm{W}^{1,\infty})}\big)$ with $\tilde C$ independent of H, ε, and Ω.

In [9, Lemma 3.11], a similar estimate is proved, with the major difference that the constant involved in the estimate depends on the Poincaré constant. In the proof of Lemma 5.2, the need of the Poincaré inequality is avoided thanks to the definition of the new elliptic projection $\pi_H \bar u$ in (5.17).

The following lemma is proved in [9].

LEMMA 5.3. *The following estimate holds for $\zeta_H = u_H - \pi_H \bar u$,*

$$\begin{aligned}\|\partial_t \zeta_H\|_{\mathrm{L}^\infty(\mathrm{L}^2)} + \|\nabla_x \zeta_H\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \leq C\Big(&\mathrm{e}_{\mathrm{H}^1}^{\mathrm{data}} + \|\eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \|\partial_t \eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \\ &+ \varepsilon^{-2}\|I_H \eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)} + \varepsilon^{-2}\|I_H \partial_t^2 \eta\|_{\mathrm{L}^\infty(\mathrm{H}^1)}\Big),\end{aligned} \tag{5.19}$$

where $\mathrm{e}_{\mathrm{H}^1}^{\mathrm{data}} = |g^0 - I_H g^0|_{\mathrm{H}^1} + \|g^1 - I_H g^1\|_{\mathrm{H}^1}$ and C is independent of H, ε and Ω.

The splitting (5.16) combined with Lemmas 5.2 and 5.3 proves Theorem 5.1.

6. Numerical experiments. In this section, we illustrate numerically our main result. We consider a one-dimensional example and compare the heterogeneous solution with several effective solutions of the family $\mathcal{E}$, the homogenized solution, and the approximation of the FE-HMM-L.

Let us fix the initial data and the right hand side for the test problem as $g_0(x) = e^{-20x^2}$, $g_1 = 0$, $f = 0$, and consider the locally periodic tensor given by

$$a\big(x, \tfrac{x}{\varepsilon}\big) = \tfrac{249}{419} + \tfrac{1}{6}\sin(2\pi x) + \tfrac{1}{6}\sin\big(2\pi \tfrac{x}{\varepsilon}\big), \tag{6.1}$$

with $\varepsilon = 1/20$. We compute explicitly $a^0(x) = 1/\big\langle 1/a(x,\cdot)\big\rangle_Y$ and $\chi(x,y) = a^0(x)\int^y 1/a(x,z)\,\mathrm{d}z - y + C_0(x)$, where C_0 ensures that $\langle \chi(x)\rangle_Y = 0$ $\forall x \in \Omega$. We verify that $\int_\Omega \sqrt{a^0(x)}\,\mathrm{d}x \approx 3/4$. We let $T^\varepsilon = \varepsilon^{-2} = 400$, and compare u^ε, u^0 and effective solutions $\tilde u_r$ in the family, where the subscript r specifies the dependence of $\tilde u$ on the parameter r in (5.3). For the waves not to reach the boundary, we consider the pseudoinfinite domain $\Omega = (-301, 301)$. To approximate u^ε, we use a spectral method on a grid of size $h_{\mathrm{ref}} = \varepsilon/25$ and the leap frog scheme for the time integration with time step $\Delta t = h_{\mathrm{ref}}/50$. To approximate u^0 and $\tilde u_r$, the same methods are used with $h = \varepsilon/4$ and $\Delta t = h/50$. Note that the second order ODE obtained after the space discretization of $\tilde u_r$ is implicit. To solve it, we use a gradient method at each time iteration of the leap frog scheme. In Figure 6.1, we display the frontal right going wave of u^ε, u^0, and $\tilde u_r$ for several values of $r \in [0, 0.1]$ at $t = \varepsilon^{-2} = 400$. We observe that the macroscopic behavior of u^ε is not well described by u^0. On the contrary, $\tilde u_r$ describes well u^ε for every value of the parameter r, as predicted by Theorem 2.3. Let us now compare the L^2 error between $u^\varepsilon(t)$ and $u^0(t)$, $\tilde u_r(t)$. We denote the normalized error as $\mathrm{err}(v)(t) = \|(u^\varepsilon - v)(t)\|_{\mathrm{L}^2(\Omega)}/\|u^\varepsilon(t)\|_{\mathrm{L}^2(\Omega)}$. In Figure 6.2, we display $\mathrm{err}(u^0)(t)$ and $\mathrm{err}(\tilde u_r)(t)$. We note that the error of the homogenized solution increases quickly with respect to t, confirming that at long times u^0 does not describe well u^ε. Next, we see that the error of $\tilde u_r$ increases notably as r increases. As Figure 6.1 showed, the frontal wave is well captured for all the values of r, hence the error for the large values of r is located elsewhere. In fact, we verify that away from the frontal wave $\tilde u_r$ drives away from u^ε as r increases. Hence, we conclude that the smaller r is the more accurate $\tilde u_r$ is.

Next, we compute the approximation provided by the FE-HMM-L, denoted by u_H, for the model problem. We let the macro FE degree be $\ell = 3$, the micro FE degree be $q = 1$, and $\delta = \varepsilon$. Referring to (5.14), we set $H = \varepsilon$ and $h = \varepsilon^{5/2}$. In Figure 6.1, we observe that the approximation u_H approximates accurately the macroscopic behavior of u^ε. In particular, u_H captures accurately the long time dispersive effects of u^ε.

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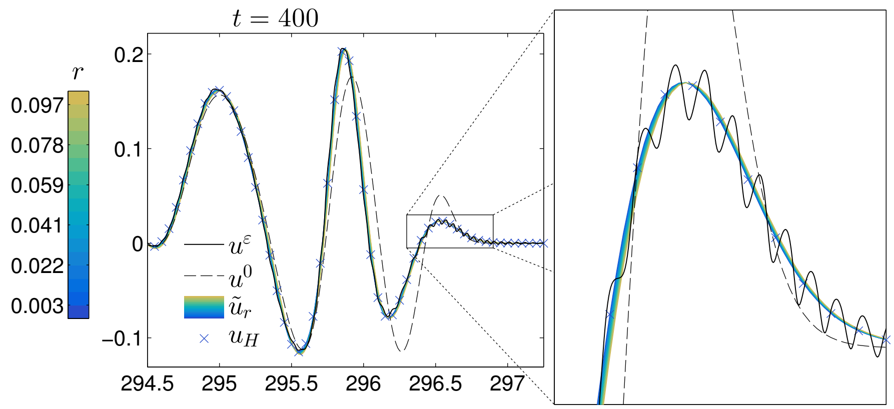

FIG. 6.1. *Comparison between the frontal waves of u^ε, u^0, $\tilde{u}_r$ and u_H at time $t = 400$ and zoom on $x \in [296.3, 296.9]$.*

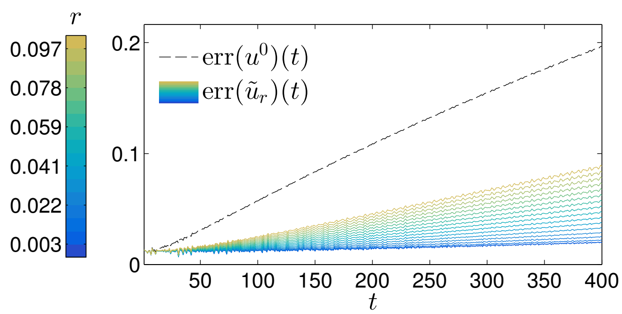

FIG. 6.2. *Comparison of the normalized L^2 error between u^ε and u^0, $\tilde{u}_r$ on the time interval $[0, 400]$.*

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