On the different convex hulls of sets involving singular values

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We give a representation formula for the convex, polyconvex and rank one convex hulls of a
set of $n \times n$ matrices with prescribed singular values.

1. Introduction

Let $\xi \in \mathbb{R}^{n \times n}$ and denote by $0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \leq \ldots \leq \lambda_n(\xi)$ the singular values of
the matrix $\xi$ (i.e. the eigenvalues of $(\xi^t \xi)^{1/2}$; this implies in particular that
$|\xi|^2 = \sum_{i=1}^{n} [\lambda_i(\xi)]^2$ and $|\det \xi| = \prod_{i=1}^{n} [\lambda_i(\xi)]$. Let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ and

$$E = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = a_i, \; i = 1, \ldots, n \}. \quad (1.1)$$

The main results of this article (cf. Theorem 3.1) are that

$$coE = \left\{ \xi \in \mathbb{R}^{n \times n} : \sum_{i=v}^{n} \lambda_i(\xi) \leq \sum_{i=v}^{n} a_i, \; v = 1, \ldots, n \right\}, \quad (1.2)$$

$$PcoE = RcoE = \left\{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=v}^{n} \lambda_i(\xi) \leq \prod_{i=v}^{n} a_i, \; v = 1, \ldots, n \right\}, \quad (1.3)$$

where $coE$ denotes the convex hull of $E$, and $PcoE$ (respectively $RcoE$) the polyconvex
(respectively the rank one convex) hull of $E$. The first notion corresponds to the
classical one (cf. [9]) while the two others will be defined in Section 2.

It is interesting to note that, if $a_1 = a_2 = \ldots = a_n$, then it turns out that

$$coE = PcoE = RcoE = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_n(\xi) \leq a_n \}$$

as already observed in [4, 6]. The case where the $a_i$ are not all equal is more involved
and has already been considered in [5, 7] when $n = 2$.

An important application of the above representations is for attainment results in
problems of the calculus of variations. A direct consequence of the results of [7] (in
particular Theorems 6.1 and 6.4) leads to the following existence theorem: let $\Omega \subset \mathbb{R}^n$
be an open set, \( a_i : \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, n \) be continuous functions satisfying
\[
0 < c \leq a_1(x, s) \leq \cdots \leq a_n(x, s)
\]
for every \((x, s) \in \bar{\Omega} \times \mathbb{R}^n\) and let \( \varphi \in C^1(\bar{\Omega}; \mathbb{R}^n) \) satisfy
\[
\prod_{i=v}^{n} \lambda_i(D\varphi(x)) < \prod_{i=v}^{n} a_i(x, \varphi(x)), \quad x \in \Omega, \quad v = 1, \ldots, n,
\]
in particular \( \varphi \equiv 0 \); then there exists \( u \in W^{1,\infty}(\Omega; \mathbb{R}^n) \) such that
\[
\begin{cases}
\lambda_i(Du(x)) = a_i(x, u(x)), & \text{a.e. } x \in \Omega, \quad i = 1, \ldots, n \\
u(x) = \varphi(x), & x \in \partial \Omega.
\end{cases}
\]

2. The different convex hulls

Before proceeding with the proofs of our main results, we introduce the following definition and properties (cf. [7] for more details).

**Definition 2.1.** Let \( E \subseteq \mathbb{R}^{m \times n} \) and
\[
F_E = \{ f : \mathbb{R}^{m \times n} \to \mathbb{R} \cup \{ +\infty \}, f|_E = 0 \}.
\]

Define
\[
coE = \{ \xi \in \mathbb{R}^{m \times n} : f(\xi) \leq 0, \forall f \in F_E, f \text{ convex} \},
\]
called the convex hull of \( E \);
\[
PcoE = \{ \xi \in \mathbb{R}^{m \times n} : f(\xi) \leq 0, \forall f \in F_E, f \text{ polyconvex} \},
\]
called the polyconvex hull of \( E \);
\[
RcoE = \{ \xi \in \mathbb{R}^{m \times n} : f(\xi) \leq 0, \forall f \in F_E, f \text{ rank one convex} \},
\]
called the rank one convex hull of \( E \).

**Remark 2.2.** The first one corresponds to the classical definition of convex hull (cf. [9]).

From the above definition, we can easily deduce the following propositions:

**Proposition 2.3.** Let \( E \subseteq \mathbb{R}^{m \times n} \); then
\[
E \subseteq RcoE \subseteq PcoE \subseteq coE.
\]

**Proposition 2.4.** Let \( E \subseteq \mathbb{R}^{m \times n} \) and define by induction
\[
R_0coE = E,
\]
\[
R_{i+1}coE = \{ \xi \in \mathbb{R}^{m \times n} : \xi = tA + (1-t)B, t \in (0, 1), A, B \in R_i coE, \text{ rank } \{ A - B \} = 1 \}.
\]
Then \( RcoE = \bigcup_{i \in \mathbb{N}} R_i coE \).

**Remark 2.5.** We can observe that the above proposition is a weaker version of the result obtained in the characterisation of convex and polyconvex hulls. For example, using Carathéodory’s Theorem, we have (cf. [9]):
\[
coE = \left\{ \xi \in \mathbb{R}^{m \times n} : \xi = \sum_{i=1}^{mn+1} t_i \xi_i, \xi_i \in E, t_i \geq 0, \text{ with } \sum_{i=1}^{mn+1} t_i = 1 \right\}.
\]
PROPOSITION 2.6. Let \( 0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \leq \ldots \leq \lambda_n(\xi) \) be the singular values of the matrix \( \xi \in \mathbb{R}^{n \times n} \). Then

(i) \( \xi \mapsto \Sigma_{i=1}^n \lambda_i(\xi) \) is a convex function, for every \( v = 1, \ldots, n \);
(ii) \( \xi \mapsto \Pi_{i=1}^n \lambda_i(\xi) \) is a polyconvex function, for every \( v = 1, \ldots, n \).

For a proof of the first result, we refer to [2, 3, 8]; for the last one, see [2] and [1], when \( n = 2 \) and \( n = 3 \) (the general case follows similarly).

3. The main results

In this section we will proceed with the proof of the main result of this article:

THEOREM 3.1. Let \( \xi \in \mathbb{R}^{n \times n} \) and denote by \( 0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \leq \ldots \leq \lambda_n(\xi) \) the singular values of the matrix \( \xi \). Let \( 0 < a_1 \leq a_2 \leq \ldots \leq a_n \),

\[ E = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = a_i, \ i = 1, \ldots, n \} \]

Then:

(i) \( \text{co}E = \{ \xi \in \mathbb{R}^{n \times n} : \Sigma_{i=1}^n \lambda_i(\xi) \leq \Sigma_{i=1}^n a_i, \ \forall \ \nu = 1, \ldots, n \} \};
(ii) \( \text{Pco}E = \text{Rco}E = \{ \xi \in \mathbb{R}^{n \times n} : \Pi_{i=1}^n \lambda_i(\xi) \leq \Pi_{i=1}^n a_i, \ \forall \ \nu = 1, \ldots, n \} \};
(iii) \( \text{intRco}E = \{ \xi \in \mathbb{R}^{n \times n} : \Pi_{i=1}^n \lambda_i(\xi) < \Pi_{i=1}^n a_i, \ \forall \ \nu = 1, \ldots, n \} \}.

REMARK 3.2. When \( n = 2 \) and \( E = \{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_1(\xi) = a_1, \lambda_2(\xi) = a_2 \} \), the theorem reads as

\[ \text{co}E = \{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_2(\xi) \leq a_2, \lambda_1(\xi) + \lambda_2(\xi) \leq a_1 + a_2 \} \]

and

\[ \text{Pco}E = \text{Rco}E = \{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_2(\xi) \leq a_2, \lambda_1(\xi) \cdot \lambda_2(\xi) \leq a_1 \cdot a_2 \} \}

Proof of Theorem 3.1(i). Let \( K = \{ \xi \in \mathbb{R}^{n \times n} : \Sigma_{i=1}^n \lambda_i(\xi) \leq \Sigma_{i=1}^n a_i, \ \forall \ \nu = 1, \ldots, n \} \}.

We show that \( \text{co}E = K \). We divide the proof into two steps.

Step 1. \( \text{co}E \subset K \). The inclusion \( \text{co}E \subset K \) is easy. In fact, \( E \subset K \) and from Proposition 2.6, the functions \( \xi \mapsto \Sigma_{i=1}^n \lambda_i(\xi) \) are convex. Therefore \( K \) is convex and hence \( \text{co}E \subset K \).

Step 2. \( K \subset \text{co}E \). Let \( \xi \in K \); we will prove that \( \xi \) can be expressed as a convex combination of elements of \( E \), i.e. \( \xi \in \text{co}E \).

Since the functions \( \xi \mapsto \lambda_i(\xi) \) are invariant by orthogonal transformations, we can assume, without loss of generality, that

\[ \xi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \]

with \( 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \) and \( \Sigma_{i=1}^n x_i \leq \Sigma_{i=1}^n a_i, \ \forall \ \nu = 1, \ldots, n \). We proceed by induction. We start with the proof in dimension \( n = 2 \).

(i) \( n = 2 \). We subdivide this case into two parts:

(a) \( x_1 \leq a_1 \) and, since \( \xi \in K \), then \( x_2 \leq a_2 \) and \( x_1 + x_2 \leq a_1 + a_2 \). Since \( -a_1 \leq x_1 \leq a_1 \), then \( x_1 = ta_1 + (1 - t)(-a_1) \) with \( t = (x_1 + a_1)/2a_1 \). We can write:

\[ \xi = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} + (1 - t) \begin{pmatrix} -a_1 \\ 0 \\ 0 \end{pmatrix}. \]

(3.1)
We proceed similarly for \( x_2 \), i.e. \( x_2 = sa_2 + (1-s)(-a_2) \), where \( s = (x_2 + a_2)/2a_2 \). Thus we obtain

\[
\begin{pmatrix}
\pm a_1 & 0 \\
0 & x_2
\end{pmatrix} = s \begin{pmatrix}
\pm a_1 & 0 \\
0 & +a_2
\end{pmatrix} + (1-s) \begin{pmatrix}
\pm a_1 & 0 \\
0 & -a_2
\end{pmatrix}.
\]

Combining (3.1) and (3.2), we get that

\[
\xi = \begin{pmatrix}
x_1 & 0 \\
0 & x_2
\end{pmatrix} = \sum_{i=1}^{l} t_i \xi_i,
\]

with \( \lambda_1(\xi_i) = a_1, \lambda_2(\xi_i) = a_2 \) (i.e. \( \xi_i \in E \)). Therefore

\[
\xi \in coE.
\]

(b) \( x_1 \geq a_1 \), i.e. since \( \xi \in K \), \( a_1 \leq x_1 \leq x_2 \leq a_2 \) and \( x_1 + x_2 \leq a_1 + a_2 \). This implies that

\[
a_1 \leq x_1 \leq a_1 + a_2 - x_2.
\]

In this case we just interpolate \( x_1 \) between \( a_1 \) and \( a_1 + a_2 - x_2 \), i.e.

\[
x_1 = ta_1 + (1-t)(a_1 + a_2 - x_2),
\]

which implies that

\[
\xi = \begin{pmatrix}
x_1 & 0 \\
0 & x_2
\end{pmatrix} = t \begin{pmatrix}
a_1 & 0 \\
0 & x_2
\end{pmatrix} + (1-t) \begin{pmatrix}
a_1 + a_2 - x_2 & 0 \\
0 & 0
\end{pmatrix}.
\]

The first matrix is treated in case (a). For the second matrix, we interpolate \( x_2 \) between \( a_1 \) and \( a_2 \), i.e. \( x_2 = sa_2 + (1-s)a_1 \), to obtain

\[
\begin{pmatrix}
a_1 + a_2 - x_2 & 0 \\
0 & x_2
\end{pmatrix} = s \begin{pmatrix}
a_1 & 0 \\
0 & a_2
\end{pmatrix} + (1-s) \begin{pmatrix}
a_2 & 0 \\
0 & a_1
\end{pmatrix}.
\]

Combining (3.3) and (3.4), we have proved that

\[
\xi = \sum_{i=1}^{l} t_i \xi_i,
\]

with \( \lambda_1(\xi_i) = a_1, \lambda_2(\xi_i) = a_2 \) (i.e. \( \xi_i \in E \)). Therefore \( \xi \in coE \). In conclusion, we have obtained, for \( n = 2 \), that

\[
K \subset coE.
\]

(ii) \( n > 2 \). We suppose that the result has been established up to \( n-1 \), i.e. every \( \xi \) such that \( \sum_{i=1}^{n-1} \phi_i(\xi) \leq \sum_{i=1}^{n-1} A_i \), \( v = 1, 2, \ldots, n-1 \) (i.e. \( \xi \in K \)) can be expressed as a convex combination of elements of \( \{ \xi \in R^{(n-1) \times (n-1)} : \phi_i(\xi) = A_i, i = 1, \ldots, n-1 \} \), i.e.

\[
\xi = \sum_{\mu=1}^{l} t_{\mu} \xi_{\mu},
\]

with \( \xi_{\mu} \) such that \( \lambda_i(\xi_{\mu}) = a_i, i = 1, 2, \ldots, (n-1) \). We divide the proof into five parts:

Part 1. \( 0 \leq x_1 \leq x_2 \leq x_1 + x_2 \leq a_2 \). Note that these conditions imply that \( x_1 + x_2 \leq a_1 + a_2 \) and \( x_2 \leq a_2 \). We can therefore apply the case \( n = 2 \) to \( \{x_1, x_2\} \) and to \( \{a_1, a_2\} \). We then use the hypothesis of induction on \( \{x_3, \ldots, x_n\} \) and on \( \{a_3, \ldots, a_n\} \). Combining these two decompositions, we get the result, i.e. \( \xi \in coE \).
Part 2. $0 \leq x_1 \leq x_2 \leq a_2 \leq x_1 + x_2$. We can write

$$
\bar{\xi} = \begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix}
= \frac{1}{2} \begin{pmatrix}
    x_1 & \lambda \\
    \lambda & x_2 \\
    \vdots & \vdots \\
    x_n
\end{pmatrix}
+ \frac{1}{2} \begin{pmatrix}
    x_1 & -\lambda \\
    -\lambda & x_2 \\
    \vdots & \vdots \\
    x_n
\end{pmatrix}
= \frac{1}{2} A_+ + \frac{1}{2} A_-,
$$

where we have chosen

$$
\lambda^2 = (x_2 - a_2)(x_1 - a_2).
$$

Note that by hypothesis ($x_1 \leq x_2 \leq a_2$) the right-hand side is positive. The choice of $\lambda$ allows us to find $O_\pm$, $O'_\pm \in O(n)$ such that

$$
O_\pm A_\pm O'_\pm = \begin{pmatrix}
    a_2 \\
    x_1 + x_2 - a_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{pmatrix}.
$$

We next apply the hypothesis of induction to

$$
\{y_1 = x_1 + x_2 - a_2, y_2 = x_3, \ldots, y_{n-1} = x_n\}
$$

and to

$$
\{b_1 = a_1, b_2 = a_3, \ldots, b_{n-1} = a_n\}.
$$

To do this, we first observe that

$$
0 \leq y_1 = x_1 + x_2 - a_2 \leq x_1 \leq x_3 = y_2 \leq y_3 \leq \ldots \leq y_{n-1}
$$

and

1. if $v \geq 2$, then $\Sigma_{i=1}^{n-1} y_i = \Sigma_{i=1}^{n+1} x_i \leq \Sigma_{i=1}^{n+1} a_i = \Sigma_{i=1}^{n-1} b_i$;
2. if $v = 1$, then $\Sigma_{i=1}^{n-1} y_i = -a_2 + \Sigma_{i=1}^{n-1} x_i \leq -a_2 + \Sigma_{i=1}^{n-1} a_i = \Sigma_{i=1}^{n-1} b_i$.

We can therefore deduce (by hypothesis of induction) that

$$
\begin{pmatrix}
    a_2 \\
    x_1 + x_2 - a_2 \\
    x_3 \\
    \vdots \\
    x_n
\end{pmatrix} \in coE.
$$

Since $coE$ is invariant up to orthogonal transformations, we obtain that

$$
A_\pm = \begin{pmatrix}
    x_1 & \pm \lambda \\
    \pm \lambda & x_2 \\
    \vdots & \vdots \\
    x_n
\end{pmatrix} \in coE,
$$

(3.6)
which leads, combining (3.5) and (3.6), to
\[ \zeta \in coE, \]
which is the claimed result.

Part 3. \( x_{n-1} \geq a_{n-1} \). We write
\[
\zeta = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \lambda + \frac{1}{2} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} - \lambda
\]
\[ = \frac{1}{2} A_+ + \frac{1}{2} A_-, \tag{3.7} \]
where we have chosen
\[ \lambda^2 = (x_n - a_{n-1})(x_{n-1} - a_{n-1}). \]

Note that by hypothesis \( (x_n \geq x_{n-1} \geq a_{n-1}) \) the right-hand side is positive. As above, the choice of \( \lambda \) leads to the existence of \( O_\pm, O'_\pm \in O(n) \) such that
\[ O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-2} \\ x_n + x_{n-1} - a_{n-1} \\ a_{n-1} \end{pmatrix}. \]

We next apply the hypothesis of induction to
\[ \{ y_1 = x_1, \ldots, y_{n-2} = x_{n-2}, y_{n-1} = x_n + x_{n-1} - a_{n-1} \} \]
and to
\[ \{ b_1 = a_1, \ldots, b_{n-2} = a_{n-2}, b_{n-1} = a_{n} \}. \]

To do this, we can observe that
\[ 0 \leq y_1 \leq \ldots \leq y_{n-2} = x_{n-2} \leq x_n \leq x_n + x_{n-1} - a_{n-1} = y_{n-1}. \]

By hypothesis and since \( \zeta \in K \), we have:
(1) if \( v = n - 1 \), \( y_{n-1} = x_n + x_{n-1} - a_{n-1} \leq a_n \);
(2) if \( 1 \leq v \leq n - 2 \),
\[ \sum_{i=v}^{n-1} y_i = x_n + x_{n-1} - a_{n-1} + \sum_{i=v}^{n-2} x_i = -a_{n-1} + \sum_{i=v}^{n} x_i \]
\[ \leq -a_{n-1} + \sum_{i=v}^{n} a_i = a_n + \sum_{i=v}^{n-2} a_i = \sum_{i=v}^{n-1} b_i. \]

We can therefore deduce by hypothesis of induction and by invariance of \( coE \) under orthogonal transformations that
\[ A_\pm \in coE, \]
which combined with (3.7) lead to
\[ \zeta \in \text{co}E. \]

**Part 4.** \[ a_2 \leq x_2 \leq \ldots \leq x_{n-1} \leq a_{n-1}. \] Note that this case occurs only if \( n \geq 4 \). We first observe that we can therefore find \( k \in \{2, \ldots, n-2\} \) such that
\[ a_k \leq x_k \leq x_{k+1} \leq a_{k+1}. \] (3.8)
Hence we can write
\[ \zeta = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \lambda \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{2} \mathcal{A}_+ + \frac{1}{2} \mathcal{A}_- \]
(3.9)
where we have chosen
\[ \lambda^2 = (x_k - b)(x_{k+1} - b) \] (3.10)
with \( b = a_k \) (Part 4.1) or \( b = a_{k+1} \) (Part 4.2). Note that, from the above assumption (3.8), the right-hand side is positive in both cases.

**Part 4.1.**
\[ \begin{cases} a_k \leq x_k \leq x_{k+1} \leq a_{k+1} \\ x_k + x_{k+1} + \sum_{i=v+1}^{n} x_i \leq a_k + \sum_{i=v}^{n} a_i, & v = k+2, \ldots, n \end{cases} \]
(with the convention \( \sum_{i=n+1}^{n} x_i = 0 \)).

**Part 4.2.**
\[ \begin{cases} a_k \leq x_k \leq x_{k+1} \leq a_{k+1} \\ \sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, & \mu = 1, \ldots, k-1. \end{cases} \]

Before proceeding with the study of the above cases, we show that Part 4.1 and Part 4.2 cover all possibilities. In fact, if \( 0 \leq x_1 \leq \ldots \leq x_n \) and if \( \Sigma_{i=v}^{n} x_i \leq \Sigma_{i=v}^{n} a_i, \ v = 1, \ldots, n \), then at least one of the following sets of inequalities holds:
\[ x_k + x_{k+1} + \sum_{i=v+1}^{n} x_i \leq a_k + \sum_{i=v}^{n} a_i, \ v = k+2, \ldots, n; \]
\[ \sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i \leq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, \ \mu = 1, \ldots, k-1. \]

We proceed by contradiction and we assume that there exists \( v \in \{k+2, \ldots, n\} \) and
\( \mu \in \{1, \ldots, k - 1\} \) such that
\[
x_k + x_{k+1} + \sum_{i=\mu+1}^{n} x_i > a_k + \sum_{i=\mu}^{n} a_i,
\]
\[
\sum_{i=\mu}^{k-1} x_i + \sum_{i=k+2}^{n} x_i > \sum_{i=\mu+1}^{k} a_i + \sum_{i=\mu+1}^{n} a_i.
\]
Summing up these two inequalities and using the assumptions, we get
\[
\sum_{i=\mu}^{n} a_i + \sum_{i=v+1}^{n} a_i \geq \sum_{i=\mu}^{n} x_i + \sum_{i=v+1}^{n} x_i > a_k - a_{k+1} + \sum_{i=\mu+1}^{n} a_i + \sum_{i=v}^{n} a_i
\]
i.e.
\[
a_\mu + a_{k+1} > a_k + a_v.
\]
However, \( \mu \in \{1, \ldots, k - 1\} \), hence \( a_\mu \leq a_k \) and \( v \in \{k + 2, \ldots, n\} \), therefore \( a_v \geq a_{k+1} \).
We therefore get
\[
a_k + a_{k+1} \geq a_\mu + a_{k+1} > a_k + a_v \geq a_k + a_{k+1},
\]
which is the claimed contradiction. In conclusion, Part 4.1 and Part 4.2 cover all possibilities. We now separately study these two cases:

Part 4.1. \( \begin{cases} a_k \leq x_k \leq x_{k+1} \leq a_{k+1} \\ x_k + x_{k+1} + \sum_{i=v+1}^{n} x_i \leq a_k + \sum_{i=v}^{n} a_i, \quad v = k + 2, \ldots, n \end{cases} \)
(with the convention \( \sum_{i=v+1}^{n} x_i = 0 \)). We choose here \( b = a_k \) in (3.9) and (3.10). We can, as above, find \( O_\pm, O'_\pm \in O(n) \) such that
\[
O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{k-1} \\ x_k & & x_k + x_{k+1} - a_k \\ & \ddots & \\ & & x_{k+1} \end{pmatrix}.
\]
We apply the hypothesis of induction to
\[
\{y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, y_k = x_k + x_{k+1} - a_k, y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_n\}
\]
and to
\[
\{b_1 = a_1, \ldots, b_{k-1} = a_{k-1}, b_k = a_{k+1}, \ldots, b_{n-1} = a_n\}.
\]
Observe that, since \( a_k \leq x_k \), then \( 0 \leq y_1 \leq \ldots \leq y_{k-1} = x_{k-1} \leq x_k + x_{k+1} - a_k = y_k \). On the contrary, \emph{a priori}, we cannot compare \( y_k \) to \( y_{k+1} \leq \ldots \leq y_{n-1} \). We next verify the hypothesis of induction.

(1) Let \( v = n - 1 \). We must show that \( y_{n-1} = x_n \leq b_{n-1} = a_n \) and \( y_k \leq b_{n-1} = a_n \).
The first inequality is valid by assumption, while the second is also true since it is equivalent to \( x_k + x_{k+1} \leq a_k + a_n \) which is the assumption of Part 4.1 with \( v = n \).
(2) Let \( n - 2 \geq v \geq k + 1 \). We have again by hypothesis of Part 4.1 and since \( \xi \in K \)

\[
\begin{align*}
\sum_{i=v}^{n-1} y_i &= \sum_{i=v}^{n} x_i \leq \sum_{i=v+1}^{n} a_i = \sum_{i=v+1}^{n-1} b_i \\
y_k + \sum_{i=v+1}^{n-1} y_i &= x_k + x_{k+1} - a_k + \sum_{i=v+2}^{n} x_i \leq \sum_{i=v+1}^{n} a_i = \sum_{i=v}^{n-1} b_i.
\end{align*}
\]

(3) If \( k \geq v \geq 1 \),

\[
\sum_{i=v}^{n-1} y_i = \sum_{i=v}^{k-1} y_i + \sum_{i=k+1}^{n-1} y_i = \sum_{i=v}^{k-1} x_i + \sum_{i=k}^{n-1} x_i - a_k \leq \sum_{i=v}^{n-1} a_i - a_k = \sum_{i=v}^{n-1} b_i.
\]

Therefore we can apply the hypothesis of induction and the invariance of \( coE \) under orthogonal transformations to get

\[
A_{\pm} \in coE.
\] (3.11)

Combining (3.9) and (3.11), we indeed get that

\[
\xi \in coE.
\]

Part 4.2.

\[
\begin{align*}
\left\{ a_k \leq x_k \leq x_{k+1} \leq a_{k+1}, \\
\sum_{i=\mu}^{k-1} x_i + \sum_{i=k+1}^{n} x_i &\leq \sum_{i=\mu+1}^{k} a_i + \sum_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k - 1.
\end{align*}
\]

We choose here \( b = a_{k+1} \) in (3.9) and (3.10). We can, as above, find \( O_{\pm}, O'_{\pm} \in O(n) \) such that

\[
O_{\pm} A_{\pm} O'_{\pm} = \begin{bmatrix}
x_1 & \cdots & x_{k-1} \\
x_k + x_{k+1} - a_{k+1} & a_{k+1} \\
& \ddots & \\
x_{k+2} & & \ddots & \cdots & \cdots & x_n
\end{bmatrix}.
\]

We apply the hypothesis of induction to

\[
\{ y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, y_k = x_k + x_{k+1} - a_{k+1}, y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_n \}
\]

and to

\[
\{ b_1 = a_1, \ldots, b_{k-1} = a_{k-1}, b_k = a_k, b_{k+1} = a_{k+2}, \ldots, b_{n-1} = a_n \}.
\]

Observe that, since \( x_{k+1} \leq a_{k+1} \), we have \( y_k = x_k + x_{k+1} - a_{k+1} \leq x_k \leq x_{k+2} = y_{k+1} \leq \cdots \leq y_{n-1} \). On the contrary, \( a \text{ priori}, \) we cannot compare \( y_k \) to \( 0 \leq y_1 \leq \cdots \leq y_{k-1} \). We next verify the hypothesis of induction. Since \( \xi \in K \) and by assumption of Part 4.2, we get:

(1) if \( v \geq k + 1 \),

\[
\Sigma_{i=v}^{n-1} y_i = \Sigma_{i=v+1}^{n} x_i \leq \Sigma_{i=v+1}^{n} a_i = \Sigma_{i=v}^{n-1} b_i;
\]

(2) if \( v \leq k + 1 \),
(2) if \( v = k \),
\[
\begin{align*}
\sum_{i=k}^{n-1} y_i &= -a_{k+1} + \sum_{i=k}^{n} x_i \
y_{k-1} + \sum_{i=k+1}^{n-1} y_i &= x_{k-1} + \sum_{i=k+2}^{n} x_i \\
q_{k+1} &= \sum_{i=k}^{n-1} b_i,
\end{align*}
\]

(3) if \( k - 1 \geq v \geq 1 \),
\[
\begin{align*}
\sum_{i=v}^{n-1} y_i &= -a_{k+1} + \sum_{i=v}^{n} x_i \
y_{k-1} + \sum_{i=v+1}^{n-1} y_i &= x_{k-1} + \sum_{i=v+2}^{n} x_i \\
q_{k+1} &= \sum_{i=v}^{n-1} b_i.
\end{align*}
\]

We can therefore apply the hypothesis of induction to obtain
\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_{k-1} \\
x_k + x_{k+1} - a_{k+1} \\
q_{k+1} \\
x_{k+2} \\
\vdots \\
x_n
\end{pmatrix} \in \text{co}E.
\]

The invariance under orthogonal transformations leads immediately to
\[
A_{\pm} \in \text{co}E. \tag{3.12}
\]

Combining (3.9) and (3.12), we have indeed obtained
\[
\xi \in \text{co}E.
\]

This achieves the proof of Step 2, i.e. \( K \subset \text{co}E \), and thus part (i) of the theorem.

Proof of Theorem 3.1(ii). Let \( X = \{ \xi \in \mathbb{R}^{n \times n} : \Pi_{i=v}^{n} \lambda_i(\xi) \leq \Pi_{i=v}^{n} a_i, \ v = 1, \ldots, n \} \). We prove that \( X = \text{RcoE} \). We divide the proof into two steps.

Step 1. \( \text{RcoE} \subset X \). Observe that \( E \subset X \) and, from Proposition 2.6, the functions \( \xi \rightarrow \Pi_{i=v}^{n} \lambda_i(\xi) \), \( v = 1, \ldots, n \) are polyconvex (and hence rank one convex). Therefore we deduce that \( X \) is polyconvex and hence
\[
\text{RcoE} \subset \text{PcoE} \subset X.
\]

Step 2. \( X \subset \text{RcoE} \). Let \( \xi \in X \); we will prove that \( \xi \in \text{RcoE} \). Since the functions \( \xi \rightarrow \lambda_i(\xi) \) are invariant by orthogonal transformations, we can assume, without loss
of generality, that

$$\xi = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$ and $\Pi_{i=1}^n x_i \leq \Pi_{i=1}^n a_i$, $v = 1, \ldots, n$.

We show the result by induction. We start with the proof in dimension $n = 2$. Note that the proof of this case is simpler than the one in [6].

(i) $n = 2$. We write

$$\xi = \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 \\ 0 \\ -x_2 \end{pmatrix} = \frac{1}{2} A_+ + \frac{1}{2} A_- \quad (3.13)$$

(observe that $rank \{A_+ - A_- \} \leq 1$) and we choose

$$\lambda^2 = \frac{(a_1^2 - x_1^2)(a_2^2 - x_2^2)}{a_2^2}.$$

Note that the right-hand side is positive by assumption ($0 \leq x_1 \leq x_2 \leq a_2$). This leads to

$$\lambda_1(A_+) = \frac{x_1 x_2}{a_2}, \quad \lambda_2(A_+) = a_2.$$

Therefore $\exists O_\pm, O'_\pm \in O(2)$ such that

$$O_\pm A_\pm O'_\pm = \begin{pmatrix} x_1 x_2 \\ a_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a_2 \end{pmatrix}.$$

However, we have

$$\begin{pmatrix} x_1 x_2 \\ a_2 \\ 0 \end{pmatrix} = \left( \frac{1}{2} + \frac{x_1 x_2}{2a_1 a_2} \right) \begin{pmatrix} a_1 \\ 0 \\ a_2 \end{pmatrix} + \left( \frac{1}{2} - \frac{x_1 x_2}{2a_1 a_2} \right) \begin{pmatrix} -a_1 \\ 0 \\ a_2 \end{pmatrix}$$

and hence

$$\begin{pmatrix} x_1 x_2 \\ a_2 \\ 0 \end{pmatrix} \in R_1 c o E \subset R c o E.$$

Since $R c o E$ is invariant up to orthogonal transformations, we deduce that

$$A_\pm = \begin{pmatrix} x_1 \\ 0 \\ \pm \lambda x_2 \end{pmatrix} \in R c o E. \quad (3.14)$$
Finally, combining (3.13) and (3.14), we obtain that

\[ \xi = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \in RcoE, \]

which is the claimed result.

(ii) \( n > 2 \). We divide this case into four parts.

Part 1. \( x_2 \leq a_2 \). We write

\[ \xi = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ \lambda \\ 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 \\ -\lambda \\ 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \]

\[ = \frac{1}{2} A_+ + \frac{1}{2} A_- \tag{3.15} \]

(observe that \( \text{rank} \{A_+ - A_-\} \leq 1 \) and we define \( \lambda \) by:

\[ \lambda^2 = \frac{(a_2^2 - x_2^2)(a_2^2 - x_1^2)}{a_2^2}. \]

Note that the right-hand side is positive by assumption \((0 \leq x_1 \leq x_2 \leq a_2)\). The choice of \( \lambda \) (as in the case \( n = 2 \)) leads to the existence of \( O_\pm, O'_\pm \in O(n) \) such that

\[ O_\pm A_\pm O'_\pm = \begin{pmatrix} a_2 \\ x_1 x_2 \\ a_2 \\ \vdots \\ x_n \end{pmatrix}. \]

We apply the hypothesis of induction to

\[ \left\{ y_1 = \frac{x_1 x_2}{a_2}, y_2 = x_3, \ldots, y_{n-1} = x_n \right\} \]

and to

\[ \left\{ b_1 = a_1, b_2 = a_3, \ldots, b_{n-1} = a_n \right\}. \]

Note that, since \( x_2 \leq a_2 \), then \( 0 \leq y_1 \leq \ldots \leq y_{n-1} \).

We have to show that \( \Pi_i^{n-1} y_i \leq \Pi_i^{n-1} b_i, \ v = 1, \ldots, n-1 \).

(1) By assumption, if \( v \geq 2 \), we have \( \Pi_i^{n-1} y_i = \Pi_i^{n-1} x_i = \Pi_i^{n-1} a_i = \Pi_i^{n-1} b_i \).

(2) If \( v = 1 \), we have

\[ \prod_{i=1}^{n-1} y_i = \frac{x_1 x_2}{a_2} \prod_{i=3}^{n} x_i = \frac{1}{a_2} \prod_{i=1}^{n} x_i \leq \frac{1}{a_2} \prod_{i=1}^{n} a_i = a_1 \prod_{i=3}^{n} a_i = \prod_{i=1}^{n-1} b_i. \]
Therefore we can deduce that (by hypothesis of induction)

\[
\begin{pmatrix}
a_2 \\
\frac{x_1 x_2}{a_2} \\
x_3 \\
\vdots \\
x_n
\end{pmatrix} 
\in RcoE.
\]

Since \(RcoE\) is invariant up to orthogonal transformations, we obtain

\[
A_\pm = \begin{pmatrix}
x_1 \\
\pm \lambda \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} 
\in RcoE
\]

and therefore, combining (3.15) and (3.16), we get

\[
\zeta \in RcoE,
\]

which is the claimed result.

**Part 2.** \(x_{n-1} \geq a_{n-1}\). We write (as in Part 1, but interchanging the role of \((x_n, x_{n-1})\) and \((x_1, x_2)\))

\[
\zeta = \begin{pmatrix}
x_1 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
x_1 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
x_1 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix}
\]

\[
= \frac{1}{2} A_+ + \frac{1}{2} A_-
\]

(observe that \(\text{rank} \{A_+ - A_-\} \leq 1\)) and we choose \(\lambda\) to be:

\[
\lambda^2 = \frac{(x_n^2 - a_{n-1}^2)(x_{n-1}^2 - a_{n-1}^2)}{a_{n-1}^2}.
\]

Note that the right-hand side is positive by assumption \((a_{n-1} \leq x_{n-1} \leq x_n)\). As above, the choice of \(\lambda\) leads to the existence of \(O_\pm, O'_\pm \in O(n)\) such that

\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-2} \\
\frac{x_{n-1} x_n}{a_{n-1}}
\end{pmatrix}.
\]

We apply the hypothesis of induction to

\[
\begin{cases}
y_1 = x_1, \ldots, y_{n-2} = x_{n-2}, y_{n-1} = \frac{x_{n-1} x_n}{a_{n-1}}
\end{cases}
\]
and to
\[ \{b_1 = a_1, \ldots, b_{n-2} = a_{n-2}, b_{n-1} = a_n\}. \]

Note that, since \( a_{n-1} \leq x_{n-1} \), then \( 0 \leq y_1 \leq \ldots \leq y_{n-1} \).

We verify the hypothesis of induction.

(1) First, observe that \( y_{n-1} \leq b_{n-1} \) because \( x_{n-1} x_n \leq a_{n-1} a_n \).

(2) If \( 1 \leq v \leq n-1 \), then
\[
\prod_{i=v}^{n-1} y_i = \prod_{i=v}^{n-2} y_i, \quad y_{n-1} = \prod_{i=v}^{n-2} x_i, \quad \frac{x_{n-1} x_n}{a_{n-1}} = \frac{1}{n} \prod_{i=v}^{n} x_i
\]
\[
\leq \frac{1}{a_{n-1}} \prod_{i=v}^{n} a_i = \prod_{i=v}^{n-1} a_i \cdot a_n = \prod_{i=v}^{n-1} b_i.
\]

Therefore (by hypothesis of induction)
\[
\left(\begin{array}{c}
x_1 \\
x_2 \\
\vdots \\
x_{n-2} \\
x_{n-1} x_n \\
\end{array}\right) \in RcoE
\]
\[
\left(\begin{array}{c}
a_{n-1} \end{array}\right)
\]

and, since \( RcoE \) is invariant up orthogonal transformations, we obtain that \( A_\pm \in RcoE \), which combined with (3.17) leads to the claimed result,
\[ \xi \in RcoE. \]

**Part 3.** \( a_2 \leq x_2 \leq \ldots \leq x_{n-1} \leq a_{n-1} \). Note that this case occurs only if \( n \geq 4 \). We first observe that we can therefore find \( k \in \{2, \ldots, n-2\} \) such that
\[ a_k \leq x_k \leq x_{k+1} \leq a_{k+1}. \]

Hence we can write
\[
\tilde{\xi} = \left(\begin{array}{c}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{array}\right) = \frac{1}{2} A_+ + \frac{1}{2} A_-
\]
\[
= \frac{1}{2} \left(\begin{array}{cccc}
x_1 & & & \\
& \ddots & & \\
& & \lambda & \\
& & 0 & x_{k+1} \\
& & & \ddots \\
& & & 0 \\
x_n & & & \\
\end{array}\right) + \frac{1}{2} \left(\begin{array}{cccc}
x_1 & & & \\
& \ddots & & \\
& & -\lambda & \\
& & 0 & x_{k+1} \\
& & & \ddots \\
& & & 0 \\
x_n & & & \\
\end{array}\right)
\]

\[ (3.19) \]
Different convex hulls of sets

(observe that \( \text{rank} \{ A_+ - A_- \} \leq 1 \)) where \( \lambda \) is given by

\[
\lambda^2 = \frac{(b^2 - x_k^2)(b^2 - x_{k+1}^2)}{b^2},
\]

where \( b = a_k \) (Part 3.1) or \( b = a_{k+1} \) (Part 3.2). Note that, from the above assumptions (3.18), the right-hand side is positive in both cases.

**Part 3.1.**

\[
\begin{cases}
    a_k \leq x_k \leq x_{k+1} \leq a_{k+1}, \\
x_k x_{k+1} \prod_{i=v+1}^{n} x_i \leq a_k \prod_{i=v}^{n} a_i, \quad v = k + 2, \ldots, n
\end{cases}
\]

(with the convention \( \prod_{i=n+1}^{n} x_i = 1 \)).

**Part 3.2.**

\[
\begin{cases}
    a_k \leq x_k \leq x_{k+1} \leq a_{k+1}, \\
    \prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k-1.
\end{cases}
\]

Before proceeding with the study of the above cases, we show that Part 3.1 and Part 3.2 cover all possibilities. In fact, if \( 0 \leq x_1 \leq \ldots \leq x_n \) and if \( \prod_{i=v}^{n} x_i \leq \prod_{i=v}^{n} a_i, \quad v = 1, \ldots, n, \) then at least one of the following sets of inequalities holds:

\[
x_k x_{k+1} \prod_{i=v+1}^{n} x_i \leq a_k \prod_{i=v}^{n} a_i, \quad v = k + 2, \ldots, n;
\]

\[
\prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k-1.
\]

We proceed by contradiction and we assume that there exist \( v \in \{ k+2, \ldots, n \} \) and \( \mu \in \{ 1, \ldots, k-1 \} \) such that

\[
x_k x_{k+1} \prod_{i=v+1}^{n} x_i > a_k \prod_{i=v}^{n} a_i,
\]

\[
\prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i > \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i.
\]

Multiplying together the two inequalities and using the assumptions, we deduce that

\[
\prod_{i=\mu}^{n} a_i \cdot \prod_{i=v+1}^{n} a_i \geq \prod_{i=\mu}^{k} x_i \cdot \prod_{i=v+1}^{n} x_i > a_k \prod_{i=v}^{n} a_i \cdot \prod_{i=k+2}^{n} a_i,
\]

i.e.

\[
a_\mu \prod_{i=\mu+1}^{n} a_i \cdot \prod_{i=v+1}^{n} a_i > a_k \prod_{i=\mu+1}^{n} a_i \cdot \prod_{i=k+2}^{n} a_i,
\]

therefore

\[
a_\mu a_{k+1} > a_k a_v.
\]

However, \( \mu \in \{ 1, \ldots, k-1 \} \), hence \( a_\mu \leq a_k \) and \( v \in \{ k+2, \ldots, n \} \), therefore \( a_v \geq a_{k+1} \).

We therefore get

\[
a_k a_{k+1} \geq a_\mu a_{k+1} > a_k a_v \geq a_k a_{k+1},
\]

which is the claimed contradiction. In conclusion, Part 3.1 and Part 3.2 cover all
possibilities. We now study these two cases separately.

Part 3.1 \[
\begin{cases}
 a_k \leq x_k \leq x_{k+1} \leq a_{k+1}, \\
 x_kx_{k+1} \prod_{i=\nu+1}^{n} x_i \leq a_k \prod_{i=\nu}^{n} a_i \quad \nu = k + 2, \ldots, n
\end{cases}
\]
(with the convention $\prod_{i=n+1}^{n} x_i = 1$). We choose here $b = a_k$ in (3.19) and (3.20); therefore we can find $O_\pm, O'_\pm \in O(n)$ such that

\[
O_\pm A_\pm O'_\pm = \begin{pmatrix}
 x_1 \\
 \vdots \\
 x_{k-1} \\
 \frac{x_kx_{k+1}}{a_k} \\
 a_k \\
 x_{k+2} \\
 \vdots \\
 x_n
\end{pmatrix},
\]

where we recall that

\[
A_\pm = \begin{pmatrix}
 x_1 \\
 \vdots \\
 x_k \\
 \pm \lambda \\
 0 \\
 x_{k+1} \\
 \vdots \\
 x_n
\end{pmatrix}.
\]

We apply the hypothesis of induction to

\[
\{y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, y_k = \frac{x_kx_{k+1}}{a_k}, y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_{n}\}
\]

and to

\[
\{b_1 = a_1, \ldots, b_{k-1} = a_{k-1}, b_k = a_{k+1}, \ldots, b_{n-1} = a_n\}.
\]

Observe that, since $a_k \leq x_k$, then $0 \leq y_1 \leq \cdots \leq y_{k-1} \leq y_k$. On the contrary, *a priori*, we cannot compare $y_k$ to $y_{k+1} \leq \cdots \leq y_{n-1}$. We next verify the hypothesis of induction.

(1) We must show that $x_n = y_{n-1} \leq b_{n-1} = a_n$ and $y_k \leq b_{n-1} = a_n$.

The first inequality is verified by assumption and the second is also verified by the assumption of Part 3.1 with $\nu = n$. The assumption $\xi \in X$ and that of Part 3.1 again ensure that

(2) if $n - 1 \geq \nu \geq k + 1$,

\[
\begin{cases}
 \prod_{i=\nu}^{n-1} y_i = \prod_{i=\nu+1}^{n} x_i \leq a_i = \prod_{i=\nu+1}^{n} b_i, \\
y_{k+1} \prod_{i=\nu+1}^{n} y_i = \frac{x_kx_{k+1}}{a_k}. \prod_{i=\nu+2}^{n} x_i \leq a_i = \prod_{i=\nu+1}^{n} b_i.
\end{cases}
\]

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(3) If \( k \geq v \geq 1 \),
\[
\prod_{i=v}^{n-1} y_i = \prod_{i=v}^{k-1} y_i \cdot \prod_{i=k+1}^{n-1} y_i = \frac{1}{a_k} \prod_{i=v}^{n} x_i \leq \frac{1}{a_k} \prod_{i=v}^{n} a_i
\]
\[
= \prod_{i=v}^{k-1} a_i \cdot \prod_{i=k+1}^{n} a_i = \prod_{i=v}^{n-1} b_i.
\]
Therefore we can apply the assumption of induction and deduce that
\[
\begin{pmatrix}
\vdots \\
x_k \\
x_{k+1} = \frac{x_k x_{k+1}}{a_k}
\end{pmatrix} \in RcoE.
\]
As above, we get that
\[
A_\pm \in RcoE
\]
and, finally, combining (3.19) and (3.21), we obtain the claimed result:
\[
\xi \in RcoE.
\]

Part 3.2
\[
\begin{cases}
  a_k \leq x_k \leq x_{k+1} \leq a_{k+1}, \\
  \prod_{i=\mu}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leq \prod_{i=\mu+1}^{k} a_i \cdot \prod_{i=k+2}^{n} a_i, \quad \mu = 1, \ldots, k-1.
\end{cases}
\]
We choose here \( b = a_{k+1} \) in (3.19) and (3.20); therefore we can find \( O_\pm, O'_\pm \in O(n) \) such that
\[
\begin{pmatrix}
\vdots \\
x_k \\
x_{k+1} = \frac{x_k x_{k+1}}{a_k}
\end{pmatrix}.
\]
We have to prove the hypothesis of induction for
\[
\begin{cases}
y_1 = x_1, \ldots, y_{k-1} = x_{k-1}, y_k = \frac{x_k x_{k+1}}{a_k+1}, y_{k+1} = x_{k+2}, \ldots, y_{n-1} = x_n
\end{cases}
\]
and for
\[ \{b_1 = a_1, \ldots, b_k = a_k, b_{k+1} = a_{k+2}, \ldots, b_{n-1} = a_n\}. \]

Observe that, since \( x_{k+1} \leq a_{k+1} \), then \( y_k \leq y_{k+1} \leq \ldots \leq y_{n-1} \). On the contrary, \emph{a priori}, we cannot compare \( y_k \) to \( y_1 \leq \ldots \leq y_{k-1} \). We verify the hypothesis of induction. From the assumption \( \xi \in X \) and from that of Part 3.2 we can write:

1. if \( v \geq k + 1 \), then \( \Pi_{i=v}^{n-1} y_i = \Pi_{i=v+1}^n x_i \leq \Pi_{i=v+1}^n a_i = \Pi_{i=v}^{n-1} b_i; \)
2. if \( v = k \), then
\[
\begin{cases}
\prod_{i=k}^{n-1} y_i = \prod_{i=k}^{n} x_i \leq \prod_{i=k}^{n} a_i = \prod_{i=k+2}^{n} b_i = \prod_{i=k}^{n-1} b_i, \\
y_{k-1} \prod_{i=k+1}^{n-1} y_i = x_{k-1} \prod_{i=k+2}^{n} x_i \leq \prod_{i=k+2}^{n} a_i = \prod_{i=k+1}^{n-1} b_i; \\
\end{cases}
\]
3. if \( k - 1 \geq v \geq 1 \), then
\[
\begin{cases}
\prod_{i=v}^{n-1} y_i = \prod_{i=v}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i = \prod_{i=v}^{k-1} a_i \cdot \prod_{i=k+2}^{n} a_i = \prod_{i=v}^{n-1} b_i, \\
\prod_{i=v+1}^{n-1} y_i = \prod_{i=v}^{k-1} x_i \cdot \prod_{i=k+2}^{n} x_i \leq \prod_{i=v}^{k-1} a_i \cdot \prod_{i=k+2}^{n} a_i = \prod_{i=v}^{n-1} b_i. \\
\end{cases}
\]

We can apply the hypothesis of induction and deduce that
\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_{k-1} \\
x_k x_{k+1} \over a_{k+1} \\
a_{k+1} \\
x_{k+2} \\
\vdots \\
x_n
\end{pmatrix} \in RcoE.
\]

Since \( RcoE \) is invariant up the orthogonal transformations, we can obtain that
\[ A_{\pm} \in RcoE. \quad (3.22) \]

Finally, combining (3.19) and (3.22), we can write \( \xi \in RcoE \). In conclusion, we have obtained the claimed result: \( X \subset RcoE. \)
Proof of Theorem 3.1(iii). Let $Y = \{ \xi \in \mathbb{R}^{n \times n} : \Pi_{i=v}^{n} \lambda_i(\xi) < \Pi_{i=v}^{n} a_i, \quad v = 1, \ldots, n \}$. We show that $\text{int } R.co.E = Y$. We divide the proof into two steps.

**Step 1.** $Y \subset \text{int } R.co.E$, since by continuity $Y$ is open and, by (ii), $Y \subset R.co.E$.

**Step 2.** $\text{int } R.co.E \subset Y$. So let $\xi \in \text{int } R.co.E$; we can therefore find $\varepsilon$ sufficiently small so that $B_\varepsilon(\xi) \subset R.co.E$ (where $B_\varepsilon(\xi)$ denotes the ball centred at $\xi$ and of radius $\varepsilon$). Let $R, R'$ be orthogonal matrices so that

$$
\begin{pmatrix}
\lambda_1(\xi) \\
\lambda_2(\xi) \\
\vdots \\
\lambda_n(\xi)
\end{pmatrix}
= R
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n
\end{pmatrix}
= R'.
$$

Define

$$
\eta = R
\begin{pmatrix}
\lambda_1(\xi) \\
\lambda_2(\xi) \\
\vdots \\
\lambda_n(\xi) + \frac{\varepsilon}{2}
\end{pmatrix}
R'.
$$

Since $|\eta - \xi| = (\varepsilon/2) < \varepsilon$, then $\eta \in R.co.E$. We then get

$$
\lambda_n(\xi) < \lambda_n(\eta) \leq a_n.
$$

Assume that $\lambda_v(\xi) \neq 0$ for every $v$; we then get for $v = 1, \ldots, n$ and with the convention $\Pi_{i=n+1}^{n} \lambda_i(\xi) = 1$,

$$
\prod_{i=v}^{n} \lambda_i(\xi) = \prod_{i=v}^{n-1} \lambda_i(\xi) \leq \prod_{i=v}^{n} \lambda_i(\eta) \leq \prod_{i=v}^{n-1} a_i \cdot a_n
$$

which implies that $\xi \in Y$.

Finally, if $\exists \tilde{v} \in \{1, \ldots, n\}$ such that $\lambda_{\tilde{v}}(\xi) = 0$, and $\lambda_{\tilde{v}+1}(\xi) > 0$, then the same argument as above is valid for $v = \tilde{v} + 1, \ldots, n$ and is trivial if $v = 1, \ldots, \tilde{v}$. We therefore also get that $\xi \in Y$. □

**Remark 3.3.** We should draw the attention to the following facts.

(1) We have privileged proofs that are as similar as possible for $co.E$ and $R.co.E$, replacing $\Sigma$ by $\Pi$. We did not succeed in doing this for the case $n = 2$.

(2) The above choice forced us, in the convex case, to consider nondiagonal (but symmetric) decompositions of the matrix $\xi$. If one insists in keeping decompositions with only diagonal matrices, then this is possible and is indeed achieved here for $n = 2$.

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References


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