Torsion classes in the cohomology of congruence subgroups

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Introduction

For any prime number $p$, let $\Gamma_{n,p}$ denote the congruence subgroup of $SL_n(\mathbb{Z})$ of level $p$, i.e. the kernel of the surjective homomorphism $f_p: SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p)$ induced by the reduction mod $p$ ($\mathbb{F}_p$ is the field with $p$ elements). We define

$$\Gamma_p := \lim_{\longrightarrow} \Gamma_{n,p}$$

using upper left inclusions $\Gamma_{n,p} \subseteq \Gamma_{n+1,p}$. Recall that the groups $\Gamma_{n,p}$ are homology stable with $M$-coefficients, for instance if $M = \mathbb{Q}$, $\mathbb{Z}[1/p]$, or $\mathbb{Z}/q$ with $q$ prime and $q \neq p$: $H_i(\Gamma_{n,p}; M) \cong H_i(\Gamma_p; M)$ for $n \geq 2i + 5$ from [7] (but the homology stability fails if $M = \mathbb{Z}$ or $\mathbb{Z}/p$).

If $p$ is an odd prime, then the group $\Gamma_p$ is torsion-free. The main objective of this paper is to detect torsion classes in the integral cohomology of $\Gamma_p$. It is actually of general interest to provide examples of torsion-free groups having torsion in their integral cohomology; this is called strange torsion in [16]. Let us mention that $H_1(\Gamma_{n,p}; \mathbb{Z})$ has been computed for $n \geq 3$ in [10]: it is a $p$-group (but does not stabilize).

In order to examine the cohomology of the congruence subgroups, our method is based on the study of the corresponding problem on the space level. As usual we denote by $BSL(\mathbb{Z})^+$ and $BSL(\mathbb{F}_p)^+$ the spaces obtained by performing the plus construction on the classifying spaces of

$$SL(\mathbb{Z}) = \lim_{\longrightarrow} SL_n(\mathbb{Z}) \text{ and } SL(\mathbb{F}_p)$$

respectively; notice that these spaces have the same (co)homology as the corresponding groups. The reduction mod $p$ induces a map $h_p: BSL(\mathbb{Z})^+ \to BSL(\mathbb{F}_p)^+$, and we call $F(p)$ its fibre and $i_p$ the inclusion map $F(p) \hookrightarrow BSL(\mathbb{Z})^+$. Consider the commutative diagram

$$
\begin{array}{ccc}
B\Gamma_p & \rightarrow & BSL(\mathbb{Z}) \overset{Bf_p}{\rightarrow} BSL(\mathbb{F}_p) \\
\downarrow \quad + & & \downarrow \quad + \\
F(p) \overset{i_p}{\rightarrow} BSL(\mathbb{Z})^+ & \overset{h_p}{\rightarrow} BSL(\mathbb{F}_p)^+ \\
\end{array}
$$

We proved in lemma 1.2 and corollary 1.3 of [2] that the map $B\Gamma_p \to F(p)$ induces an isomorphism on (co)homology with $M$-coefficients if $M$ is again $\mathbb{Q}$, $\mathbb{Z}[1/p]$, or $\mathbb{Z}/q$.  


with $q \neq p$ (this is of course also true for the plus construction). In particular, the cohomological restriction $H^*(SL(Z); M) \to H^*(\Gamma_p; M)$ may be identified with the homomorphism $i^*: H^*(BSL(Z)^*; M) \to H^*(F(p); M)$.

Our main result is given in the first section of the paper, which is devoted to mod 2 cohomology. If $p$ is a prime and $p \equiv 3 \mod 4$, we are able to compute the algebra $H^*(\Gamma_p; \mathbb{Z}/2)$. We prove for instance that the even-dimensional Stiefel–Whitney classes in $H^*(SL(Z); \mathbb{Z}/2)$ do not become trivial when restricted to $\Gamma_p$, for all primes $p \equiv 3 \mod 4$; see [11] for a more general discussion of elements in the cohomology of an arithmetic group which are not killed by passage to subgroups of finite index. Our calculation produces many 2-torsion classes in the integral cohomology of $F_p$.

In the second section we introduce a general argument which shows that for infinite loop spaces the Hurewicz homomorphism is injective on $q$-torsion elements if $q$ is a sufficiently large prime (in comparison with the dimension we are looking at). The $q$-torsion classes discovered in the algebraic $K$-theory of $\mathbb{Z}$ in [13] (where $q$ is related to the numerator of the Bernoulli numbers) survive in the homotopy groups of $F(p)$, and consequently also in the integral (co)homology of $\Gamma_p$ via the Hurewicz homomorphism, assuming $p \neq q$.

1. Mod 2 cohomology

In this section we determine the restriction homomorphism $H^*(SL(Z); \mathbb{Z}/2) \to H^*(\Gamma_p; \mathbb{Z}/2)$ for all prime numbers $p \equiv 3 \mod 4$. Let $GL(Z)$ be the infinite general linear group of $\mathbb{Z}$ and $w_i \in H^i(GL(Z); \mathbb{Z}/2)$ the $i$th Stiefel–Whitney class of the inclusion $GL(Z) \hookrightarrow GL(\mathbb{R})$ for $i \geq 1$; we shall also denote by $w_i$ the image of $w_i$ under the restriction $H^i(GL(Z); \mathbb{Z}/2) \to H^i(SL(Z); \mathbb{Z}/2)$ for $i > 1$ (the first Stiefel–Whitney class of $SL(Z)$ is zero). Let $Q_n$ denote the subgroup of diagonal matrices in $GL_n(Z)$; notice that $Q_n \cong (\mathbb{Z}/2)^n$ and $H^*(Q_n; \mathbb{Z}/2) = \mathbb{Z}/2[v_1, v_2, \ldots, v_n]$, where $deg v_j = 1$ for $1 \leq j \leq n$. As usual, write $\sigma_i$ for the $i$th elementary symmetric function of $v_1, v_2, \ldots, v_n$. Now let us consider the inclusion $\phi_n: Q_n \hookrightarrow GL_n(Z) \hookrightarrow GL(\mathbb{Z})$ and the induced homomorphism $\phi_n^*: H^*(GL(Z); \mathbb{Z}/2) \to H^*(Q_n; \mathbb{Z}/2)$.

**Lemma 1.1.** (a) For any element $\alpha \in H^*(GL(Z); \mathbb{Z}/2)$ and for each $n \geq 1$, $\phi_n^*(\alpha)$ is a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$; (b) $\phi_n^*(w_i) = \sigma_i$ for $i \geq 1$, $n \geq i$.

**Proof.** The action of the symmetric group $S_n$ on $H^*(Q_n; \mathbb{Z}/2)$ is induced by conjugation by an element of $GL_n(Z)$, therefore it is trivial on the image of $\phi_n^*$. This shows (a) and assertion (b) is proved in theorem 22.7 of [5].

**Proposition 1.2.** There is a commutative graded algebra $A$ such that

(a) $H^*(GL(Z); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3, \ldots] \otimes A$,

(b) $H^*(SL(Z); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \ldots] \otimes A$.

**Proof.** It is obvious that the classes $w_i$, for $i \geq 1$, are non-trivial and algebraically independent in $H^*(GL(Z); \mathbb{Z}/2)$ by assertion (b) of the previous lemma. On the other hand, $H^*(GL(Z); \mathbb{Z}/2)$ is the cohomology of the $H$-space $BGL(Z)^+$ and thus may be written in the following form:

$$H^*(GL(Z); \mathbb{Z}/2) = \bigotimes_{j=1}^{\infty} B_j,$$
Cohomology of congruence subgroups

where for each $j$, $B_j$ is either $\mathbb{Z}/2[x_j]$ or $\mathbb{Z}/2[x_j]/(x_j^2 = 0)$, where $e_j$ is a power of 2. It is now sufficient to prove that the $w_i$'s are not decomposable in $H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ (we then may replace some of the $x_j$'s by the $w_i$'s and deduce (a)). But the assumption that $w_i$ is decomposable would imply, after applying the homomorphism $\phi^*_p$ for some $n \geq i$, that $\sigma_j$ is a polynomial in the $\sigma_j$'s with $j < i$; this is not the case.

The second part is immediate because $BSL(\mathbb{Z})^+$ is the fundamental cover of $BGL(\mathbb{Z})^+$ and $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$.

Remark 1:3. It has been conjectured, in relation to the Quillen–Lichtenbaum conjecture, that $A$ is an exterior algebra $\Lambda(u_3, u_5, \ldots, u_{2k+1}, \ldots)$ where $\deg u_{2k+1} = 2k+1$ (cf. [8], corollary 4:3). Observe that, if $Z$ denotes the fibre of the obvious infinite loop map $\psi: BSL(\mathbb{Z})^+ \to BSO$, the Serre spectral sequence of the fibration $Z \to BSL(\mathbb{Z})^+ \to BSO$ collapses because $\psi^*: H^*(BSO; \mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ is injective (cf. [5], theorem 15:2) and one can conclude that $A \simeq H^*(Z; \mathbb{Z}/2)$. But the study of the Eilenberg–Moore spectral sequence (with rational coefficients) of this fibration shows that the rational cohomology of $Z$ is an exterior algebra with one generator in each odd dimension $\geq 3$. This exhibits non-trivial elements $u_{2k+1}$ in $A$ for all $k \geq 1$.

The mod 2 cohomology algebra of the infinite general linear group $GL(\mathbb{F}_p)$ ($p$ odd) is known (cf. [12], or §IV-8 of [9]): it is generated by classes $c_i$ and $e_i$ for $i \geq 1$, where $\deg c_i = 2i$ and $\deg e_i = 2i-1$. If $p$ is a prime $\equiv 3 \mod 4$, one has the relations

$$e_i^2 = \sum_{0 \leq j < i} c_j c_{2i-1-j}$$

and the mod 2 cohomology algebra is polynomial:

$$H^*(GL(\mathbb{F}_p); \mathbb{Z}/2) = \mathbb{Z}/2[e_1, e_2, e_3, \ldots, c_2, c_4, c_6, \ldots].$$

We write also $c_i$ (respectively $e_i$) for the restriction of $c_i$ (resp. $e_i$) to $SL(\mathbb{F}_p)$:

$$H^*(SL(\mathbb{F}_p); \mathbb{Z}/2) = \mathbb{Z}/2[e_2, e_3, \ldots, c_2, c_4, c_6, \ldots].$$

Lemma 1:4. Let $p$ be an odd prime, $g_p: GL(\mathbb{Z}) \to GL(\mathbb{F}_p)$ the homomorphism induced by reduction mod $p$ and $g_p^*: H^*(GL(\mathbb{F}_p); \mathbb{Z}/2) \to H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ the induced homomorphism. Then $g_p^*(c_i) = w_i^2$ for $i \geq 1$.

Proof. Recall that the cohomology class $c_i$ comes from the reduction mod 2 of the $i$th universal Chern class in the cohomology of $BU$, via the Brauer lift. According to [4], $g_p^*(c_i)$ is then the reduction mod 2 of the $i$th Chern class of the inclusion $GL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$; but this is $w_i^2$ by theorem 5:3 of [15] or proposition 25:6 of [5].

Lemma 1:5. If $p \equiv 3 \mod 4$ and $i \geq 1$, then

$$g_p^*(e_i) = \sum_{0 \leq j < i} w_j w_{2i-1-j} + \gamma_i,$$

where $\phi_p^*(\gamma_i) = 0$ for $n \geq 2i-1$.

Proof. We deduce from the relation (*) and the previous lemma that

$$(g_p^*(e_i))^2 = \sum_{0 \leq j < i} w_j^2 w_{2i-1-j}.$$
and consequently that
\[(\phi_n^*(g_p^*(e_i)))^2 = (\sum_{0 \leq j < i} \sigma_j \sigma_{2i-1-j})^2.\]
Since \(\phi_n^*(g_p^*(e_i))\) is a polynomial in \(\sigma_1, \sigma_2, \ldots, \sigma_n\) by Lemma 1.1, \(\phi_n^*(g_p^*(e_i))\) must be \(\sum_{0 \leq j < i} \sigma_j \sigma_{2i-1-j}\). The proof is then complete because
\[\phi_n^*(g_p^*(e_i)) = \phi_n^*(\sum_{0 \leq j < i} w_j w_{2i-1-j}).\]

Remark 1.6. The fact that \(\phi_n^* (\alpha_j) = 0\) implies that \(\gamma_i \notin \mathbb{Z}/2[w_1, w_2, w_3, \ldots]\). It follows in particular from Lemma 1.5 that \(g_p^*(e_i) \neq 0\).

Proposition 1.7. Let \(p\) be a prime congruent to 3 mod 4. Then the homomorphisms \(g_p^*: H^*(GL(F_p); \mathbb{Z}/2) \to H^*(GL(\mathbb{Z}); \mathbb{Z}/2)\) and \(f_p^*: H^*(SL(F_p); \mathbb{Z}/2) \to H^*(SL(\mathbb{Z}); \mathbb{Z}/2)\), induced by reduction mod \(p\), are injective.

Proof. Recall that \(H^*(GL(F_p); \mathbb{Z}/2) = \mathbb{Z}[e_1, e_2, e_3, \ldots, c_2, c_3, \ldots]\). We have seen that \(g_p^*(c_i)\) and \(g_p^*(e_i)\) are non-trivial in \(H^*(GL(\mathbb{Z}); \mathbb{Z}/2)\) for \(i > 1\). In order to establish the injectivity of \(g_p^*\), we must verify that there are no polynomial relations between the elements \(g_p^*(c_i)\) (i even, \(i \geq 2\)), \(g_p^*(e_i)\) (\(i \geq 1\)). But this follows after applying the homomorphism \(\phi_n^*\) (with \(n\) large enough), because we know that
\[\phi_n^*(g_p^*(c_i)) = \sigma_i^2\] and \(\phi_n^*(g_p^*(e_i)) = \sum_{0 \leq j < i} \sigma_j \sigma_{2i-1-j}\) (and that there are no polynomial relations among the \(\sigma_j\)'s). A similar proof provides the injectivity of \(f_p^*\).

Remark 1.8. This argument is not valid if \(p \equiv 1 \text{ mod } 4\); the assertion of Proposition 1.7 is actually wrong for primes \(p \equiv 1 \text{ mod } 8\) (see [1], proposition 1).

Remember that \(H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \ldots] \otimes A\) and write \(w_i(\Gamma_p)\) for the image of \(w_i\) under the restriction homomorphism \(H^i(SL(\mathbb{Z}); \mathbb{Z}/2) \to H^i(\Gamma_p; \mathbb{Z}/2), i \geq 2\).

Theorem 1.9. If \(p\) is a prime congruent to 3 mod 4, then the restriction homomorphism \(H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \to H^*(\Gamma_p; \mathbb{Z}/2)\) is surjective and its image is
\[H^*(\Gamma_p; \mathbb{Z}/2) \cong \Lambda(w_2(\Gamma_p), w_4(\Gamma_p), \ldots, w_{2k}(\Gamma_p), \ldots) \otimes A.\]

Proof. We actually work on the space level. In the fibration
\[F(p) \to BSL(\mathbb{Z})^{+} \to BSL(F_p)^{+},\]
observe that all spaces are infinite loop spaces and all maps are infinite loop maps. Therefore, since
\[h_p^*: H^*(BSL(F_p)^{+}; \mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^{+}; \mathbb{Z}/2)\]
is injective by the previous proposition, the homomorphism
\[i_p^*: H^*(BSL(\mathbb{Z})^{+}; \mathbb{Z}/2) \to H^*(F(p); \mathbb{Z}/2)\]
is surjective and the Serre spectral sequence collapses:
\[E_\infty = E_2 \cong H^*(BSL(F_p)^{+}; \mathbb{Z}/2) \otimes H^*(F(p); \mathbb{Z}/2)\]
(cf. [5], theorem 15-2). It remains to examine the kernel of \(i_p^*\).
Cohomology of congruence subgroups

Let \( i \) be an integer with \( i \geq 2 \) and \( K_i \) the subgroup of \( H^i(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \) generated by elements of the form \( xy \), where \( x \) is of positive degree in the image of \( h_p^* \) and \( y \in H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \). Clearly the kernel of \( i_p^*: H^i(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \rightarrow H^i(F(p); \mathbb{Z}/2) \) contains \( K_i \); we want to show that it is exactly \( K_i \) by checking that the dimension of this kernel (as \( \mathbb{Z}/2 \)-vector space), i.e.

\[
\dim \bigoplus_{j=1}^{i} H^j(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \otimes H^{i-j}(F(p); \mathbb{Z}/2),
\]

is less than or equal to the dimension of \( K_i \). Take a basis of

\[
\bigoplus_{j=1}^{i} H^j(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \otimes H^{i-j}(F(p); \mathbb{Z}/2)
\]

and associate to each element \( \xi \otimes \eta \) of this basis an element \( xy \) of \( K_i \) as follows: \( x := h_p^*(\xi) \) and \( y \) is a cohomology class in \( H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \) satisfying \( i_p^*(y) = \eta \); we choose \( y \) such that it is a sum of elements of the form \( w \otimes a \), where \( w \) is a square-free polynomial in the even-dimensional Stiefel-Whitney classes and \( a \in A \) (since \( i_p^* \) vanishes on the image of \( h_p^* \), Lemmas 1.4 and 1.5 prove inductively that this is possible). With this choice it is not hard to verify that these associated elements are linearly independent in \( K_i \).

Consequently

\[
H^*(F(p); \mathbb{Z}/2) \cong H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)/\text{Ker } i_p^* \
\cong H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)/(\text{Im } h_p^*) \
\cong \Lambda(w_2, w_4, \ldots, w_{2k}, \ldots) \otimes A.
\]

In particular, \( i_p^* \) does not vanish on even-dimensional Stiefel-Whitney classes or on elements of \( A \).

**Corollary 1.10.** If \( p \equiv 3 \mod 4 \), then \( H^i(\Gamma_p; \mathbb{Z}/2) \neq 0 \) for all \( i \geq 2 \).

**Proof.** The algebra \( A \) contains a non-trivial element \( u_3 \) of degree 3 (cf. Remark 1.3). For \( i \geq 2 \), define \( \omega_i \in H^i(\Gamma_p; \mathbb{Z}/2) \) to be \( w_i(\Gamma_p) \) if \( i \) is even and \( w_{i-3}(\Gamma_p) \otimes u_3 \) if \( i \) is odd; the previous theorem asserts that \( \omega_i \neq 0 \). Notice that since \( F(p) \) is simply connected \( H^1(\Gamma_p; \mathbb{Z}/2) = H^1(F(p); \mathbb{Z}/2) = 0 \).

We close this section by mentioning that the non-vanishing of the even-dimensional Stiefel-Whitney classes of \( \Gamma_p \) (for \( p \equiv 3 \mod 4 \)) produces 2-torsion classes in its integral cohomology groups in arbitrarily large dimensions. (According to the definition given in [16], \( \Gamma_p \) has very strange 2-torsion.)

**Corollary 1.11.** Let \( p \) be a prime congruent to 3 mod 4 and \( i \) an even integer \( \geq 2 \). Then \( w_i(\Gamma_p) \) detects 2-torsion in \( H^i(\Gamma_p; \mathbb{Z}) \) or in \( H^{i+1}(\Gamma_p; \mathbb{Z}) \).

**Proof.** The Stiefel-Whitney class \( w_i(\Gamma_p) \) is not the image of an element of infinite order under the reduction mod 2, because \( i \) is even and the rational cohomology of \( \Gamma_p \) is an exterior algebra generated by classes of degree 5, 9, ..., \( 4k+1 \), ... (see [2], theorem 1.4).
2. Spherical classes in the homology of infinite loop spaces

The purpose of this section is to prove the following

**Theorem 2.1.** Let $X$ be an $m$-connected infinite loop space (where $m \geq 0$), $i$ an integer greater than $m$ and $q$ a prime greater than $(i-m)/2+1$. If $\pi_i X$ contains an element $\alpha$ of order $q^r$ (with $r \geq 1$), then the image of $\alpha$ under the Hurewicz homomorphism is also of order $q^r$ in $H_i(X; \mathbb{Z})$.

This will be useful for the study of odd torsion classes in the (co)homology of congruence subgroups. The proof of Theorem 2.1 is based on the next result which follows from the discussion of the $\mathbb{Z}$-invariants of iterated loop spaces: see [3].

**Proposition 2.2.** There exist positive integers $S_j$ ($j \geq 1$) with the following property: if $X$ is an $m$-connected infinite loop space (where $m \geq 0$ and $i$ an integer greater than $m$, then there is a map $f : X \to K(\pi_i X, i)$ such that the induced homomorphism $f_* : \pi_i X \to \pi_i X$ is multiplication by a divisor of $S_{i-m}$. (These integers $S_j$ are defined in [3]; a prime number $q$ divides $S_j$ if and only if $q \leq j/2 + 1$.)

**Proof of Theorem 2.1.** Let us look at the commutative diagram induced by the map $f$ introduced in the previous proposition:

$\begin{array}{ccc}
\pi_i X & \xrightarrow{f_*} & \pi_i K(\pi_i X, i) \\
\downarrow \text{Hu} & & \downarrow \text{Hu} \\
H_i(X; \mathbb{Z}) & \xrightarrow{f_*} & H_i(K(\pi_i X, i); \mathbb{Z})
\end{array}$

($\text{Hu}$ denotes the Hurewicz homomorphism). If $\alpha$ is an element of order $q^r$ in $\pi_i X$, $f_* \circ \text{Hu}(\alpha)$ is again of order $q^r$ by Proposition 2.2. With this argument we may conclude in fact that if $\alpha$ generates a cyclic direct summand of order $q^r$ in $\pi_i X$, then the same is true for $\text{Hu}(\alpha)$ in $H_i(X; \mathbb{Z})$.

We obtain also information on the Pontryagin ring structure of $H_\ast(X; \mathbb{Z})$.

**Corollary 2.3.** Let $X$ be an $m$-connected infinite loop space (where $m \geq 0$), $i$ an even integer greater than $m$ and $q$ a prime greater than $(i-m)/2+1$. Assume that $\pi_i X$ contains a cyclic direct summand of order $q^r$ (with $r \geq 1$) generated by $\alpha$ and define $\beta := \text{Hu}(\alpha) \in H_i(X; \mathbb{Z})$. Then $\beta^k$ is a non-trivial $q$-torsion element of $H_{ki}(X; \mathbb{Z})$ for each $k$ such that $((k-1)!)_q < q^r$. (Here $(\ )_q$ denotes the $q$-primary part.)

**Proof.** Consider the composition

$\lambda : H_\ast(X; \mathbb{Z}) \to H_\ast(K(\pi_i X, i); \mathbb{Z}) \to H_\ast(K(\mathbb{Z}/q^r, i); \mathbb{Z})$

where the second arrow is induced by the projection $\pi_i X \to \mathbb{Z}/q^r$; $\lambda$ is a ring homomorphism since it is possible to choose a loop map for $f$. Define $\gamma := \lambda(\beta)$; it is a generator of $H_i(K(\mathbb{Z}/q^r, i); \mathbb{Z})$ by Theorem 2.1. The ring structure of $H_\ast(K(\mathbb{Z}/q^r, i); \mathbb{Z})$ is known (see [6]) and provides an interesting conclusion if $i$ is even: $\gamma^k$ is of order $(k!q^r)/(k!)_q$. Thus $\gamma^k \neq 0$ if $q^r > ((k-1)!)_q$ and the non-vanishing of $\beta^k$ follows from $\lambda(\beta^k) = \gamma^k$. 


Remark 2.4. If $X$ is an $m$-connected $s$-fold loop space (where $m \geq 0$, $s \geq 0$), then the above results remain valid if $i$ satisfies $m < i \leq s + 2m$ (cf. [3]).

The next assertion on the infinite loop space $BSL(Z)^+$ is given on page 290 in [13]: let $i$ be an even integer, $q$ a properly irregular prime with $q > i$ and such that $q$ divides the numerator of $B_{i}/i$ (where $B_i$ is the $i$th Bernoulli number: $B_2 = 1/6$, $B_4 = 1/30$, ...); then there exists $q$-torsion in $K_{2i-2}Z = \pi_{2i-2}BSL(Z)^+$ and in $H_{2i-2}(BSL(Z)^+; Z)$. Notice that the localization exact sequence in algebraic $K$-theory ([14], théorème 1) shows that this $q$-torsion survives into $K_{2i-2}Q$ and therefore into $H_{2i-2}(BSL(Q)^+; Z)$ according to Theorem 2.1. Observe also that Corollary 2-3 implies actually the existence of $q$-torsion in $H_{k(2i-2)}(BSL(Z)^+; Z)$ for $1 \leq k \leq q$. It is interesting to remark that the corresponding $q$-torsion classes in $H^{k(2i-2)+1}(BSL(Z)^+; Z)$ do not come from the cohomology of $BU$ via the homomorphism induced by a complex representation of $SL(Z)$, because $H^{k}(BU; Z) = 0$ if $j$ is odd.

Now let us look at the homotopy sequence of the fibration $F(p) \rightarrow BSL(Z)^+ \rightarrow BSL(F(p))$:

$$\ldots \rightarrow \pi_{2i-2}F(p) \rightarrow K_{2i-2}Z \rightarrow K_{2i-2}F(p) \rightarrow \ldots$$

Since $K_{2i-2}F(p) = 0$ from [12], the group $\pi_{2i-2}F(p)$ (which is finitely generated) also contains $q$-torsion. We apply Corollary 2-3 to the infinite loop space $F(p)$ and detect $q$-torsion in $H_{k(2i-2)}(F(p); Z)$ for $1 \leq k \leq q$. If we suppose $p \neq q$, we obtain $q$-torsion in the integral (co)homology of $\Gamma_p$ because $H_\ast(\Gamma_p; Z[1/p]) \cong H_\ast(F(p); Z[1/p])$. We obtain

**Theorem 2-5.** Let $i$ be an even integer, $q$ a properly irregular prime with $q > i$ and such that $q$ divides the numerator of $B_{i}/i$. Then there exists $q$-torsion in $H_{k(2i-2)}(\Gamma_p; Z)$ for $1 \leq k \leq q$ and all primes $p \neq q$.

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